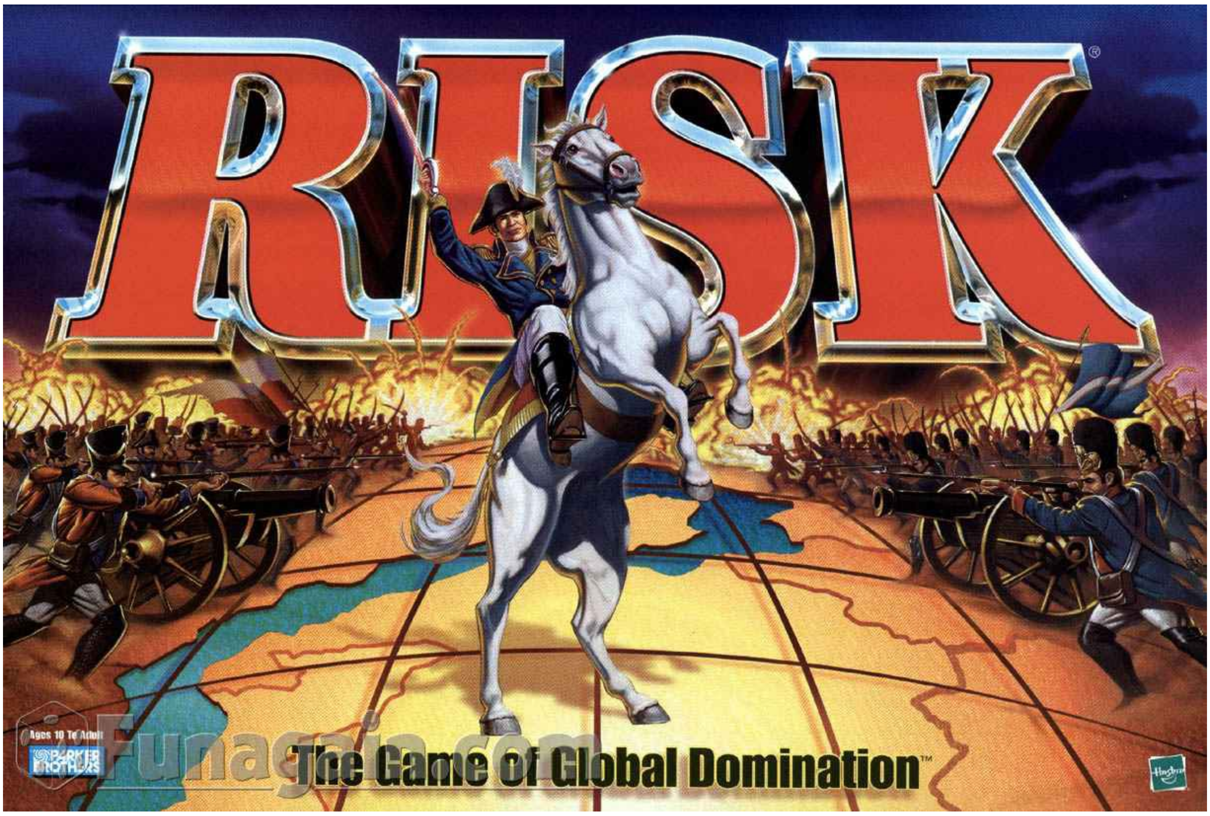


MATHS INTERNAL ASSESSMENT

Introduction

My internal assessment is to do with probability; as a kid I would always play the board game risk with my brother or friends. It is a game that is to do with strategy with the aim of conquering the world. I got the idea when we studied the probability unit earlier this year, and I thought I could apply that knowledge to find the probability behind the game Risk. This is a game that has many different courses of action and many different scenarios, this is where it gets the name Risk from, and I remember always trying new strategies to try and get the upper hand, back then it was just by guessing and instinct, but now I will try and solve the probability and figure out what is the best strategy to win in Risk.

I will try and find the probability of the attack winning in each of the situations using tree diagrams and being able to hopefully find an equation for any number of dice. I will cross reference my results with those I found on the internet to see if I get the right value and then will also create an excel file to carry out every outcome, to simulate every possible roll of dice.

The main aim of my exploration is to calculate the probability of the attack winning in all the different situations in the game.

Rules

The game consists of a world map that is divided up in to 42 “countries” for which you start off with a certain amount depending on the number of players as they are divided equally between the players, each country has an equivalent territory card. At the start you have a certain amount of “infantry” or “troops” that you can place on the country and each turn you can add more troops on to these countries which is also dependent whether or not you control any continents or have gained any territory cards. The main point of the game is attacking enemy territories, which you conquer when you defeat all the infantry they have on that country, you can attack as many times and as many territories as you like in your turn. You can only attack when you have a territory with more than one infantry as one has to stay on the territory at all times. When you attack you can attack with 1, 2 or 3 troops, the defence on the other hand regardless of the number of infantry on the territory can chose to defend with one or two troops but no more than two. Each is represented by a normal 6 sided dice (normally red dice is attacking and black is for defending but I will use yellow as attacking and red for defending) when it comes to a battle for both the attacker and defender. The rules state that the attacker can choose to attack with 1 to 3 troops and the defence can defend with up to 2 troops if already on the territory. Then both player grab the number of dice equivalent to their number of troops (each dice represents one of the troops), and roll the dice at the same time. The attack wins if their dice is larger than that of the defence. If both dice are the same then defence wins as they have territorial advantage. If you roll more than one dice you chose the larger of the two. In the case scenario of the defence is defending with two troops and the attack with two or three, then the highest rolled dice of the attack goes against the highest dice of the defence. Then the second highest of the attack goes against the second highest of the defence. Where it then follows the same rules as when it’s a simple 1vs1. When you lose a battle meaning you rolled a smaller number (or same if attack) you must remove one of your troops from the board, since each die represents a troop.

Source of photo: www.theloftlounge.ca

Main Body

There are six case scenarios in a battle which are (defence vs attack) 1vs1, 1vs2, 1vs3, 2vs1, 2vs2, and 2vs3. I will calculate each of these scenarios individually, I will also divide them into two different groups as one when the defence only has one infantry (dice) and when it has two infantry (dice).

n = number of dice

All calculations involve a normal 6 sided dice numbered 1 to 6

First I need to find the number of outcomes which are as followed:

Scenario	Possible outcomes
1vs1	36
1vs2	216
1vs3	1296
2vs1	216
2vs2	1296
2vs3	7776

The total number of outcomes is simply 6^n as every time you add an additional die there are 6 times as many possible outcomes.

1vs1

Here is a table of outcomes where the attack losing is represented by an L and a win by a W

		Defence					
		1	2	3	4	5	6
Attack	1	L	L	L	L	L	L
	2	W	L	L	L	L	L
	3	W	W	L	L	L	L
	4	W	W	W	L	L	L
	5	W	W	W	W	L	L
	6	W	W	W	W	W	L

(In the first image the defence wins in second the attack does)

If score of attack die ≤ score of defence's die

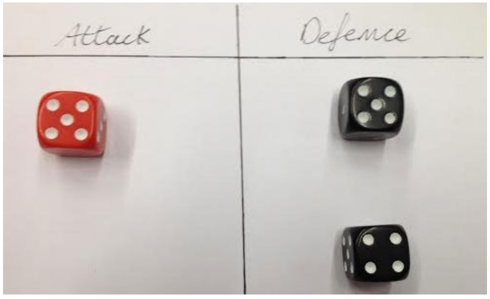
The defence wins

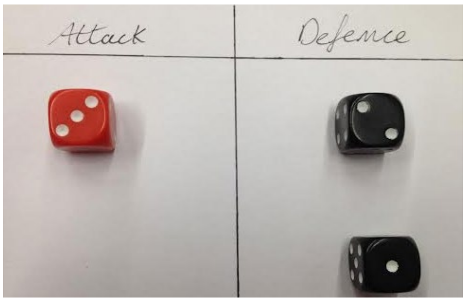
By counting out the probability of attack winning (red "w") is 15/36. The attacker wins if A > D where the probability density function (pdf) for A or D = 1, 2, 3, 4, 5, or 6 is 1/6 so from the law of total probability which is $\Pr(A) = \sum_n \Pr(A \cap Bn)$ for any avent A of the same probability space, so

Pr(A>D)= $(Pr(A > 1) \times \Pr(D = 1)) + (\Pr(A > 2) \times \Pr(D = 2)) + \cdots + (\Pr(A > 6) \times \Pr(D = 6))$

$$= \frac{1}{36} \times (5 + 4 + 3 + 2 + 1 + 0) = \frac{15}{36}$$

1vs2



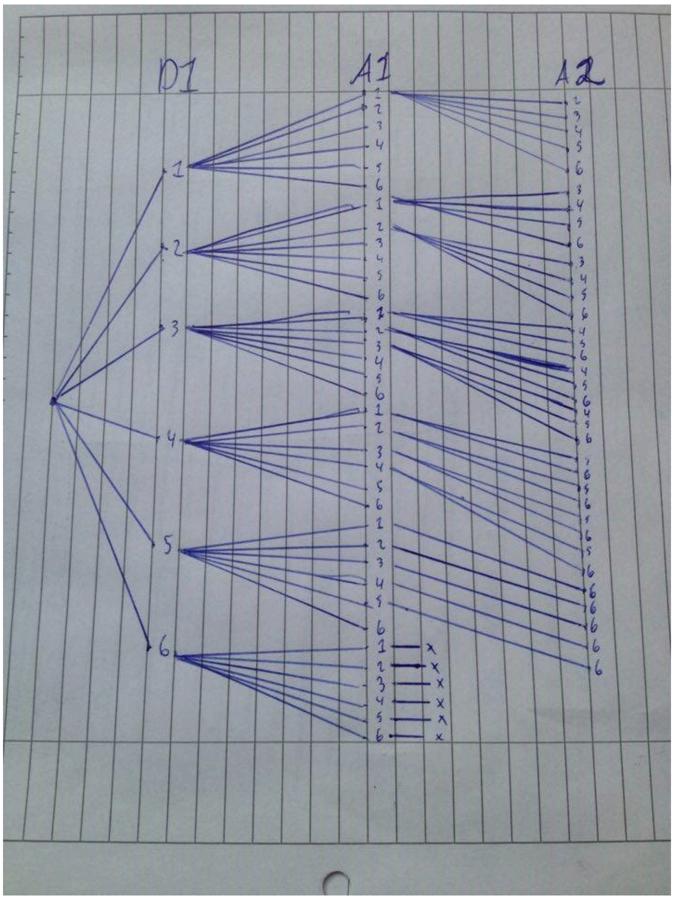


(In the first image the defence wins in the second the attack does)

I tried working out this probability similarly to the second method in 1vs1, where I believed it was like a binomial with 2 trials as attacker wins if A1 is greater than D or if A2 is greater than D, and because A1 and A2 have the same distribution. Where the probability of one of the dice beating the defenders one is 15/36. So it made sense that the probability of attack winning out of the two case scenarios is 1 – Pr(of defence winning)

$$\Pr(attack\ winning) = 1 - \left((2C0) \times \left(\frac{21}{36}\right)^2 \right) = 0.65972$$

This answer is wrong, this is why I below I tried a different method to calculate the probability:



To the left is an image of a tree diagram I made for the scenario of 1 defender (D1) 2 attackers (A1 & A2)

There is a total of 216 possible outcomes when rolling three dice as 6 x 6 x 6 = 216

The tree diagram only shows the case scenarios where the Attack wins, this is when A1 or A2 is larger than D1 (A1,A2>D1). So this is why A2 only shows the values that are larger than D1.

To calculate the total probability of Attack winning you count the number of branches in A2 and add it to the number of branches in A1 that do not have a branch to A2 and multiply that by 6 as the value of A2 does not matter. Without counting the branches for when D1 is equal to 6 as cannot be beaten.

Probability of of attackers winning $\frac{((A2\times6)+A1)}{216}$

Branches in A2 = 35

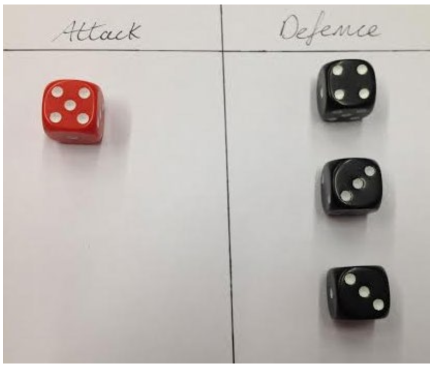
Branches in A1 = 15

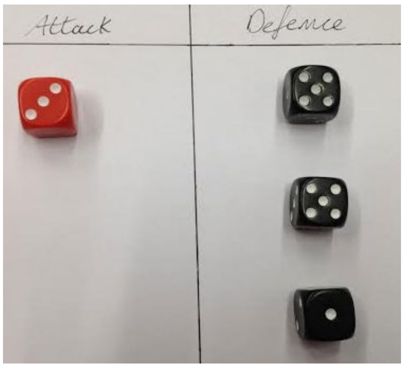
So probability of Attack winning is $\frac{(15\times6)+35}{216}=\frac{125}{216}=0.5787$

I believe my answer using my first method is wrong because It is not a binomial as you chose the largest value between the two dice (A1, A2) so they are part of the same trial.

- The reason it isn't a binomial distribution is because it doesn't follow these four requirements
- A fixed number of trials
- Each trial is independent of the others
- There are only two outcomes
- The probability of each outcome remains constant from trial to trial.

1vs3





(In the first image the attack wins in the second the defence does)

I tried to make a tree diagram but it got two complicated and too small for the piece of paper as well as time consuming as there are a total of 1296 (6^4) outcomes for 4 dice. So I will try a different method using probability and will try and find a general equation for when the defender has one dice and for n number of dice of the attacker.

Attack wins if one of the attackers dice is larger than that of defence ((A1,A2,A3)>D1)

$$\Pr(maximum\ of\ (A1,A2,A3)>D1)$$

$$=(\Pr(max(A1,A2,A3)>1)\times\Pr(D1=1))+$$

$$(\Pr(\max(A1,A2,A3)>2)\times\Pr(D1=2))+\cdots+(\Pr(\max(\mathrm{A1,A2,A3})>6)\times\Pr(\mathrm{D1}=6))$$

Which can be used to calculate the probability by

$$(1-\Pr(\max(A1,A2,A3)\leq1)\times\Pr(D1=1)+\cdots+(1-\Pr(\max(\mathrm{A1,A2,A3})\leq6)\times\Pr(\mathrm{D1}=6)$$

$$=[(1-(\Pr(A1\leq1)\times\Pr(A2\leq1)\times\Pr(A3\leq1))\times\Pr(D1=1)]+\cdots+[(1-(\Pr(A1\leq6)\times\Pr(A2\leq6)\times\Pr(A3\leq6))\times\Pr(D1=6)]$$

$$=\left[\left(1-\left(\frac{1}{6}\right)^3\right)\times\frac{1}{6}\right]+\left[\left(1-\left(\frac{2}{6}\right)^3\right)\times\frac{1}{6}\right]+\cdots+\left[\left(1-\left(\frac{6}{6}\right)^3\right)\times\frac{1}{6}\right]=\sum_{n=1}^{6}\left[1-\left(\frac{n}{6}\right)^3\right]\times\frac{1}{6}$$

$$=\left(\frac{1}{6}\right)^4\times[(6^4)-(1^3+2^3+3^3+4^3+5^3+6^3]$$

$$\frac{95}{144}=0.65972$$

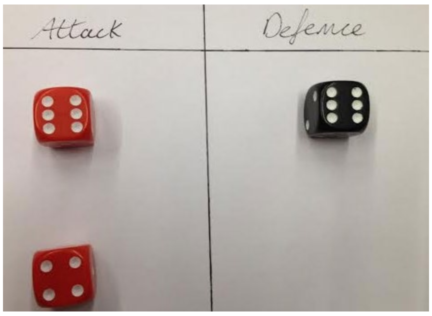
From this we can draw a general equation for n number of defence infantry against one defence. Which the probability of an attacker with n dice vs 1 defender dice is:

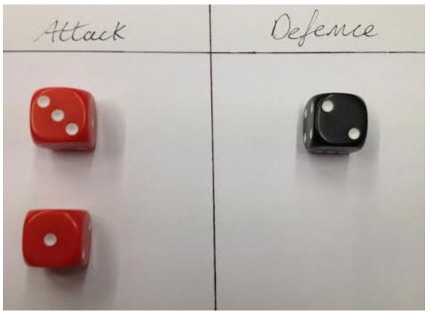
$$= \left(\frac{1}{6}\right)^{(n+1)} \times \left[\left(6^{(n+1)}\right) - (1^n + 2^n + 3^n + 4^n + 5^n + 6^n)\right]$$

We can check our answer to 1vs2 using this general formula:

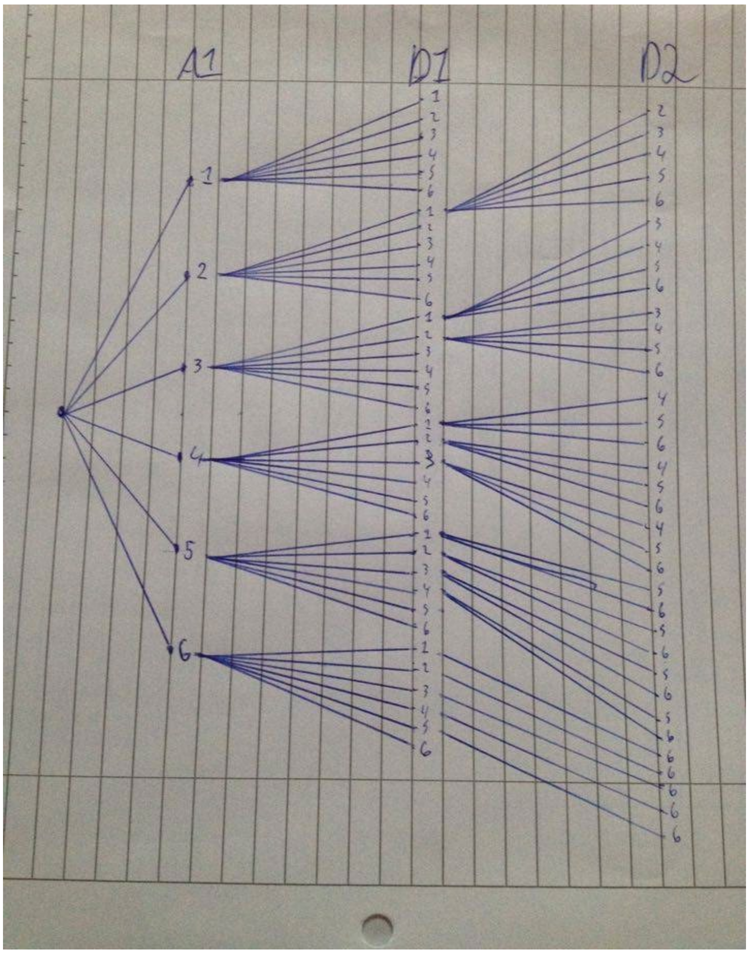
$$= \left(\frac{1}{6}\right)^{3} \times [(6^3) - (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2] = \frac{125}{216} = 0.5787$$

2vs1





(In the first image the attack loses in the second the attack wins)



For this scenario I chose to just use a tree diagram as due to the rules that when there is a battle where both players have two or more troops then both troops can be lost or where both loose one. This is why it is easier to just use a tree diagram, as well as it wouldn't fit in a general equation for n attackers against 2 defenders.

To the left is an image of the tree diagram similar to that of 1vs2 only inversed as it shows the outcomes where the defence wins

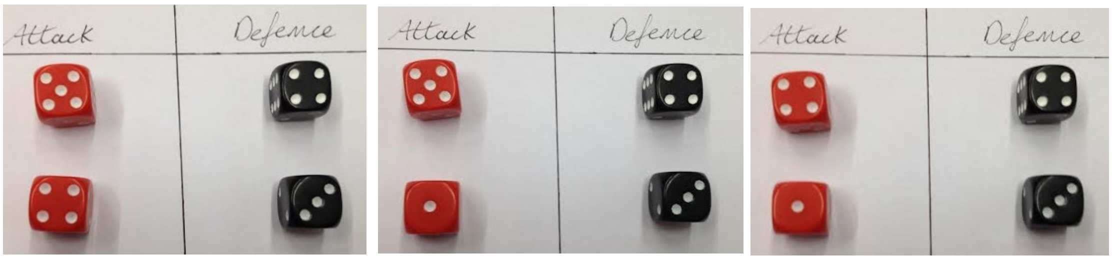
To calculate the probability of attack I will do 1- the probability of the defence winning. Or by finding the number of outcomes the defence wins from the diagram and then to find the amount of times the attack wins which is 216 – the times the defence wins.

The number of values in D2 which is 35 which we then add to the amount numbers that do not have a branch that goes on to D2 multiplied by 6 as no matter what value of D2 (which is between 1 and 6) the defence still wins. This gives 35 + 21 x 6 = 161

This is the total number of outcomes out of 216 for which the defence wins so 216 – 161 = 55 is the number of outcomes where the attack wins.

The probability the attack wins is $= \frac{55}{216} = 1 - \frac{161}{216} = 0.25463$

2vs2



(The first photo shows the attack winning both, the second shows the defence winning both, and the third is a tie where both lose one and win one)

Since both the attack and the defence have two troops, both troops are in play, this is somewhat equivalent as to having to 1vs1 battles. When attacking you can either lose both troops by rolling a lower number then your opponent. You can also keep one troop but lose another when one of your dice is higher than the other, and if both of your dice are bigger than the opponents you can keep both troops. As both players roll two dice each representing an infantry, a total of four dice is rolled. Out of the two dice you rolled you chose the larger of the two which goes against the larger of the two dice of your opponent, and then you do the same with the second larger of the dice which in this case is the smallest as well. So there are two battles where it is largest against largest and second largest against second largest. This is where I believed it would be to take it into two separate case scenarios highest scoring dice compared to lowest scoring dice.

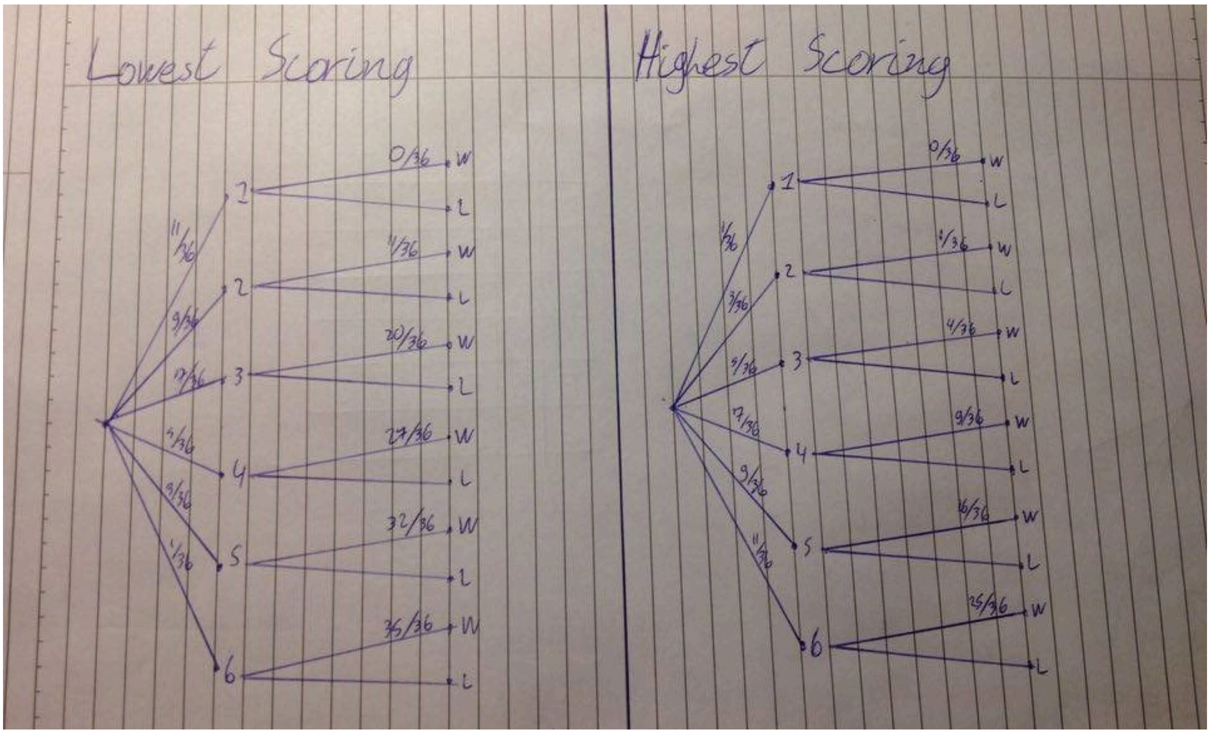
Below on the left if a grid which shows out of the two dice that a player may roll in this case scenario which of the two is largest, and below on the right is the probability for the higher or the lowest value of the two dice.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

Lowest	1	2	3	4	5	6
P(X=x)	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

Highest	1	2	3	4	5	6
P(X=x)	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Using this information I have made two tree diagrams (as shown below) for the attack winning with the lowest and highest of the two dice.



Using the tree diagram on the right to first calculate the Probability that the attack wins with the highest dice is

$$= \left(\frac{1}{36} \times \frac{0}{36}\right) + \left(\frac{3}{36} \times \frac{1}{36}\right) + \left(\frac{5}{36} \times \frac{4}{36}\right) + \left(\frac{7}{36} \times \frac{9}{36}\right) + \left(\frac{9}{36} \times \frac{16}{36}\right) + \left(\frac{11}{36} \times \frac{25}{36}\right) = \frac{505}{1296}$$

And by looking at the numerator of the first fraction which is the probability for the highest value of the two dice being k (which is either 1, 2, 3, 4, 5, and 6), it follows an arithmetic sequence of $2k - 1$ and the second numerator which is the probability that the attacker wins with that highest number follows a geometric sequence of $(k - 1)^2$. This leads to a general formula of

$$\sum_{k=1}^{6} \left(\frac{2k - 1}{36}\right) \times \left(\frac{(k - 1)^2}{36}\right)$$

But that is just half as still have to find the probability of the second largest (smallest) of the two dice wins. As or else would be just a tie and both the defender and attack lose a troop.

$$= \left(\frac{11}{36} \times \frac{0}{36}\right) + \left(\frac{9}{36} \times \frac{11}{36}\right) + \left(\frac{7}{36} \times \frac{20}{36}\right) + \left(\frac{5}{36} \times \frac{27}{36}\right) + \left(\frac{3}{36} \times \frac{32}{36}\right) + \left(\frac{1}{36} \times \frac{35}{36}\right) = \frac{505}{1296}$$

The first of the two fractions the numerator does follow an arithmetic sequence of 12-(2k-1) but the second fraction doesn't follow neither geometric nor arithmetic sequence as does not have a common ratio or difference.

From this I calculated that the probability of the attack winning both battles is

$$= \frac{505}{1296} + \frac{505}{1296} = \frac{505}{648} = 0.7793$$

This seems to be very high to me and favours the attack a lot so I decided to create an excel that would calculate every scenario and then would calculate which scenarios would the attack win both (excel file embedded in appendix). And I was correct as the probability of the attack winning is

$$= \frac{295}{1296} = 0.2276$$

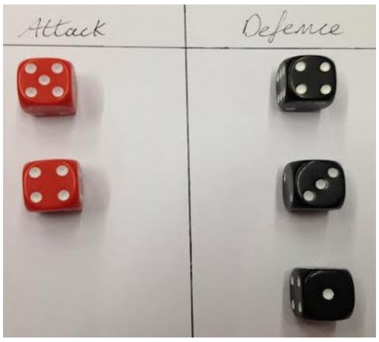
Which I believe is much more reasonable considering the game is meant to be where the odds for the attack winning and the defence are nearly similar. I believe that my error before was due to not considering that either of the two dice could be the largest, which I thought I could go around by imagining that one dice will always be bigger than the other. But then it was the same for the defender so there is a lot of outcomes I did not take into account in my calculations. The excel formula I used was

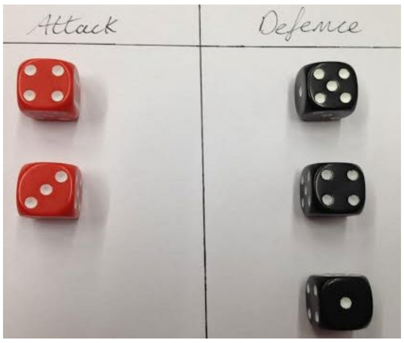
=IF(MIN($A3:$B3)>MIN(C$1:C$2),1,0)+IF(MAX($A3:$B3)>MAX(C$1:C$2),1,0)

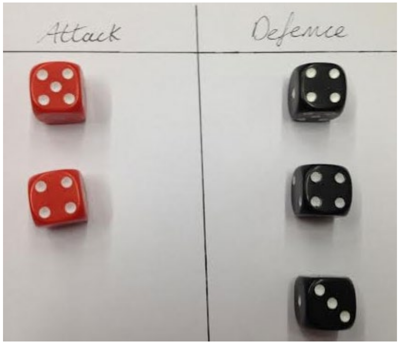
Where A3 and B3 was the values of the attackers dice and C1 and C2 where that of the defence, I then used =COUNTIF(C3:AL38,2) to calculate the number of cells that added up to 2 where the attack won both battles.

	A	B	C	D	E	F	G	H
1	ATTACK YELLOW		1	1	1	1	1	1
2	295	/1296	1	2	3	4	5	6
3	1	1	0	0	0	0	0	0
4	1	2	1	0	0	0	0	0
5	1	3	1	1	0	0	0	0
6	1	4	1	1	1	0	0	0
7	1	5	1	1	1	1	0	0
8	1	6	1	1	1	1	1	0
9	2	1	1	0	0	0	0	0
10	2	2	2	1	1	1	1	1
11	2	3	2	2	1	1	1	1
12	2	4	2	2	2	1	1	1
13	2	5	2	2	2	2	1	1
14	2	6	2	2	2	2	2	1

2vs3







The same as for 2vs2 I did not draw a tree diagram as there are 7776 (6^5) possible outcomes so I used excel to calculate the probability from all the possible outcomes I used a formula that took into account the second largest for the attack and not smallest as there was three options.

=IF(LARGE(A3:C3,2)>MIN(D$1:D$2),1,0)+IF(MAX($A3:$B3:$C3)>MAX(D$1:D$2),1,0)

Which is the same as for 2vs2 only with three dice so you chose the two largest.

This gave a probability of the attack winning both battles of

$$= \frac{2890}{7776} = \frac{1445}{3888} = 0.3717$$

	A	B	C	D	E	F	G	H	I	J
1	ATTACK YELLOW			1	1	1	1	1	1	2
2	2890	/7776		1	2	3	4	5	6	1
3	1	1	1	0	0	0	0	0	0	0
4	1	1	2	1	0	0	0	0	0	0
5	1	1	3	1	1	0	0	0	0	1
6	1	1	4	1	1	1	0	0	0	1
7	1	1	5	1	1	1	1	0	0	1
8	1	1	6	1	1	1	1	1	0	1
9	1	2	1	1	0	0	0	0	0	0
10	1	2	2	2	1	1	1	1	1	1
11	1	2	3	2	2	1	1	1	1	2
12	1	2	4	2	2	2	1	1	1	2
13	1	2	5	2	2	2	2	1	1	2
14	1	2	6	2	2	2	2	2	1	2
15	1	3	1	1	1	0	0	0	0	1
16	1	3	2	2	2	1	1	1	1	2
17	1	3	3	2	2	1	1	1	1	2

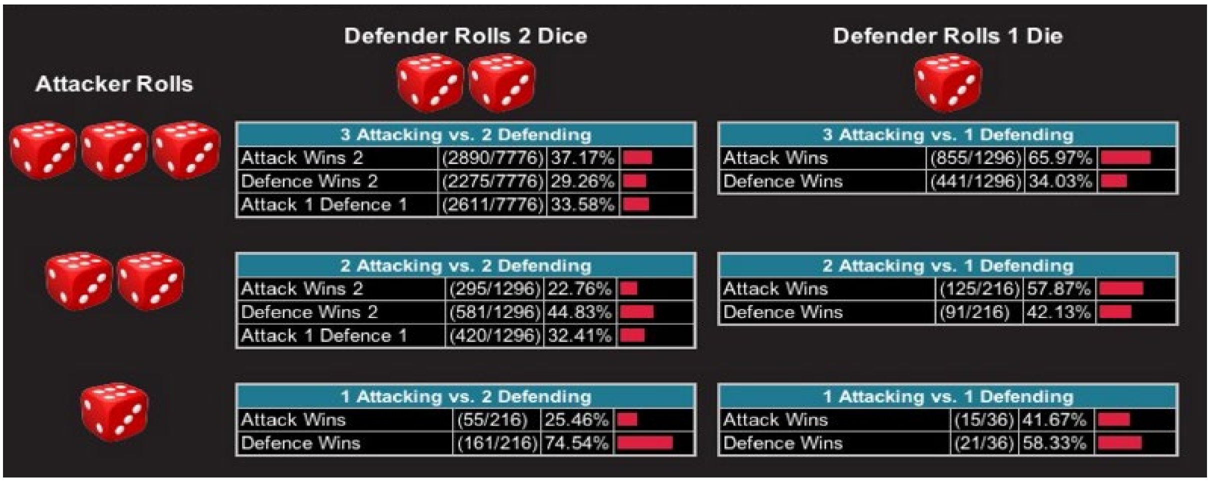
Conclusion

Scenario	Probability attack wins
1vs1	15/36
1vs2	125/216
1vs3	855/1296
2vs1	55/216
2vs2	295/1296
2vs3	2890/7776

Above is a table of my results for the probability that the attack wins.

To verify my results I searched online for the probability of different battles in risk and found a lot of websites that used programs to calculate the outcomes including simulators which gave me the idea to continue my excel for all of the different types of battles in risk. On http://www.businessinsider.com/how-to-use-math-to-win-at-the-board-game-risk-2013-7?IR=T

I found this table which showed every probability in the game which shows that my calculations where correct and that the results I gained from my excel are as well.



3 Attacking vs. 2 Defending		
Attack Wins 2	(2890/7776)	37.17%
Defence Wins 2	(2275/7776)	29.26%
Attack 1 Defence 1	(2611/7776)	33.58%

3 Attacking vs. 1 Defending		
Attack Wins	(855/1296)	65.97%
Defence Wins	(441/1296)	34.03%

2 Attacking vs. 2 Defending		
Attack Wins 2	(295/1296)	22.76%
Defence Wins 2	(581/1296)	44.83%
Attack 1 Defence 1	(420/1296)	32.41%

2 Attacking vs. 1 Defending		
Attack Wins	(125/216)	57.87%
Defence Wins	(91/216)	42.13%

1 Attacking vs. 2 Defending		
Attack Wins	(55/216)	25.46%
Defence Wins	(161/216)	74.54%

1 Attacking vs. 1 Defending		
Attack Wins	(15/36)	41.67%
Defence Wins	(21/36)	58.33%

It is clear that the more dice you are attacking with the higher your probability of winning is especially if the defence only is using one dice where you should win about 2 out of 3 times if it was a 1 on 3. Contrary it isn't wise to attack when the defender has two dice, even though attacking is part of the game and only way to conquer other territories and win the game. The odds in the game are quick even so it isn't easy to determine what the best strategy is especially when you don't know what your opponents move, but that is why risk is such a risky game. Although I think these probabilities can really help someone playing the game as it shows how its best to attack with more troops, as well as where to place your troops anticipating future attacks.

Using the tree diagrams did help me find a general formula for when there was only one defence dice and 'n' number of attack dice. But was not a suitable approach for when there where a large number of outcomes, as I already had problems when there was 216 outcomes, as spacing wasn't easy. Using excel was much easier, although working out the correct formula was not easy once it worked it was really quick as it did all the outcomes for you; this is why I used worked out every scenario using excel as well. Using excel it is very easy to extend to 4 vs 3 dice for example or other combinations, and maybe a pattern could be found that can lead to a general formula. Other way to extend the investigation would be to study different strategies as the game also involves some card probability when it comes to the territory cards.

Bibliography

Hickey, W., 2013. *Buisness insider.* [Online]
Available at: http://www.businessinsider.com/how-to-use-math-to-win-at-the-board-game-risk-2013-7?IR=T
[Accessed 12th February 2016].

minivan15, 2010. *ActuarialOutpost.* [Online]
Available at:
http://www.actuarialoutpost.com/actuarial_discussion_forum/showthread.php?t=199026&goto=nextoldest
[Accessed 12th February 2016].

Appendix

Risk.xlsx