Predicting the horizontal distance of a bouncing ball

Introduction

Dropping, or even throwing, a ball seems pretty simple. From bounce passes in basketball to the back and forth in table-tennis, the motion of a bouncing ball is an everyday occurrence. The math behind it, however, can be complex.

The mathematical aspects that I will be focusing on are calculus and kinematics. One application of calculus is differentiating, which can be used to find the rate of change, and another is integrating. In relation to motion, integration can be used to link acceleration with velocity and displacement.

'Considering how many fools can calculate, it is surprising that it should be thought either a difficult or tedious task for any other fool to learn how to master the same tricks.' – Thompson, S (1922)

Narrowing down a topic to investigate for my internal assessment was very difficult. After researching potential topics, I decided to stick with my initial idea of projectiles and ballistics primarily because I was interested in the crossover with physics and the application of calculus in the 'real world'. When looking through engineering courses at university, the math portion of the course seemed to draw heavily on calculus, which further encouraged my choice of topic.

From here, I chose to investigate the motion of a bouncing ball because it is a very common movement with a complex mathematical description, which in itself is fascinating. Yet the math could be simplified to a suitable level for this investigation. The motion of the ball also offered many possible factors to investigate, for example how the initial height would impact the motion. However I decided on the following aim.

Aim: to theoretically calculate the total horizontal distance after a single bounce of a ping-pong from being dropped, given an initial height and horizontal velocity.

This could be useful as, for a given velocity, it would then be possible to predict where an object would land after bouncing. Furthermore, calculating the horizontal distance may be useful for developing a computer game or as an illustration of engineering techniques.

Setting up the problem

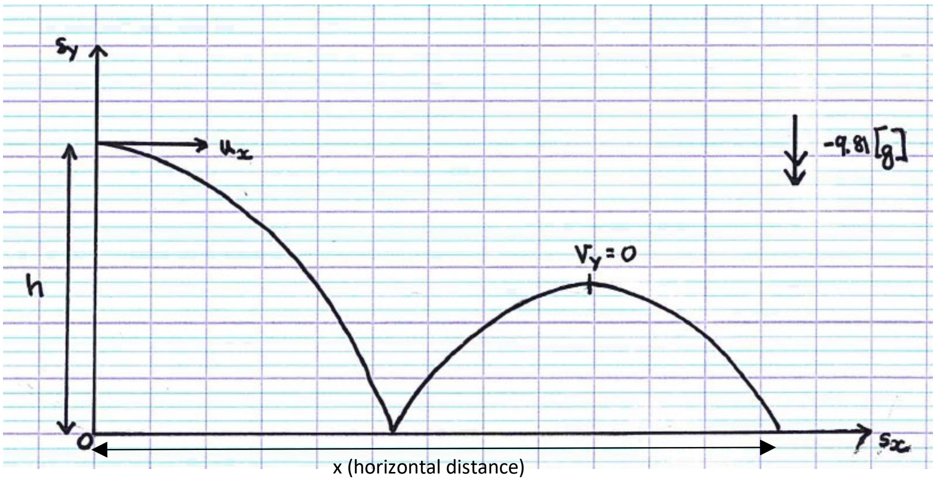


Figure 1: Sketch of problem

Variables and parameters:

h – initial drop height (m)

g – acceleration due to gravity (ms^{-2})

u – initial velocity (ms^{-1})

v – velocity (ms^{-1})

a – acceleration (ms^{-2})

s – displacement (m)

F – force (N)

m – mass (g)

k – constant of proportionality (air resistance)

p – air density (gm^{-3})

A – cross-sectional area (m^2)

C_D – drag coefficient

I initially compared a squash ball and a ping-pong ball to determine which to use. The squash ball did not bounce to a suitable height while the ping-pong ball both bounced to a suitable height and, when dropped vertically, would rebound relatively vertically within a couple of trials. As this ball was suitable, I did not look at any further balls.

Assumptions

- The ball has no spin
- During impact there is no friction between the ball and the surface
- The ball does not deform during impact.
- Though air resistance is taken into account in the equations of motion, I have not taken air resistance into account when calculating the coefficient of restitution

Deriving equations of motion

In the motion of a bouncing ball, the horizontal and vertical parts are independent and therefore can be separated. (*Allum and Talbot, 2014*)

Acceleration

Newton's second law (*Allum and Talbot, 2014*)

$$F_{net} = ma$$

Vertical Acceleration

The force of gravity acts on the ball in the vertical direction as does air resistance.

$$Air\ resistance = -kv^n$$

Where k is a constant of proportionality. I will assume air resistance is directly proportional to velocity, so the value of n=1.

$$k = \frac{1}{2} \times C_D \times \rho \times A$$

As 'k' is a constant for the same conditions, I will continue to use the term 'k' rather than the equation when deriving for simplicity and put the constants in when calculating for the specific case.

Therefore the net force is the sum of the force of gravity and air resistance, which act in opposite directions.

$$F_{net} = mg - kv$$

$$ma = mg - kv$$

$$a = g - \frac{kv}{m}$$ **Equation 1**

(*oregonstate.edu, no date*)

Horizontal acceleration

The only horizontal force on the ball is air resistance.

$$F_{net} = -kv$$

$$ma = -kv$$

$$a = \frac{-kv}{m}$$ **Equation 2**

Velocity

Vertical velocity

$$a = \frac{dv}{dt}$$

$$\therefore m\frac{dv}{dt} = mg - kv$$

This is a first order linear differential equation in the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

So

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

The integral factor is

$$I = e^{\int \frac{k}{m} dt} = e^{\frac{k}{m}t}$$

Therefore

$$e^{\frac{k}{m}t}\frac{dv}{dt} + \frac{k}{m}e^{\frac{k}{m}t}v = ge^{\frac{k}{m}t}$$

$$\frac{d}{dt}\left(e^{\frac{k}{m}t}v\right) = ge^{\frac{k}{m}t}$$

$$e^{\frac{k}{m}t}v = \int ge^{\frac{k}{m}t}\,dt$$

Integrating gives

$$e^{\frac{k}{m}t}v = \frac{mg}{k}e^{\frac{k}{m}t} + c$$

$$v = \frac{mg}{k} + ce^{-\frac{k}{m}t}$$

When t=0 there is no vertical velocity so v=0

$$0 = \frac{mg}{k} + 1c \;\; \therefore c = \frac{-mg}{k}$$

$$\therefore v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) \qquad \textbf{Equation 3}$$

<u>Horizontal velocity</u>

$$a = -\frac{kv}{m}$$

$$a = \frac{dv}{dt}$$

$$\therefore \frac{dv}{dt} = -\frac{kv}{m}$$

Using separable variables

$$\int \frac{1}{v}dv = \int -\frac{k}{m}dt$$

$$lnv = -\frac{k}{m}t + c$$

$$v = Ae^{-\frac{k}{m}t}$$

When t=0 the velocity is the initial velocity so v=u

$$u = Ae^{-\frac{k}{m}0}$$

$$u = A$$

$$\therefore v = ue^{-\frac{k}{m}t}$$ **Equation 4**

Displacement

$$s = \int v\,dt$$

<u>Vertical Displacement</u>

Integrating ‘v’ from **Equation 3** gives vertical displacement

$$v = \frac{mg}{k} - \frac{mg}{k}e^{-\frac{k}{m}t}$$

$$s = \int \frac{mg}{k}t^0 dt - \int \frac{mg}{k}e^{\frac{-k}{m}t}dt$$

$$s = \frac{mgt}{k} + \frac{m^2g}{k^2}e^{-\frac{k}{m}t} + c$$

When t=0 then the vertical displacement is the original height so s=h

$$h = 0 + \frac{m^2g}{k^2} + c$$

$$c = -\frac{m^2g}{k^2} + h$$

$$\therefore s = h + \frac{mg}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t} - \frac{m}{k}\right)$$ **Equation 5**

<u>Horizontal Displacement</u>

Integrating horizontal velocity (**Equation 4**) gives horizontal displacement

$$v = ue^{-\frac{k}{m}t}$$

$$s = \int ue^{-\frac{k}{m}t}dt$$

$$s = -\frac{mue^{-\frac{k}{m}t}}{k} + c$$

When t=0 there is no horizontal displacement so s=0

$$0 = -\frac{mu}{k} + c$$

$$c = \frac{mu}{k}$$

$$\therefore s = -\frac{mu}{k}\left(e^{-\frac{k}{m}t} - 1\right)$$ **Equation 6**

Application in specific case

Setting up the specific case

I will predict the motion of a ping-pong ball. To do this, the constants for acceleration due to gravity, mass and the constant of proportionality must be established.

Acceleration is a vector quantity. I have chosen acceleration due to gravity to be negative in the direction towards Earth. I have also chosen to record acceleration due to gravity to two decimal places. (*Allum and Talbot, 2014*)

$g = -9.81ms^{-2}$

Using a 2-decimal place balance (accurate to 0.01g), I measured the mass of the ping-pong ball by putting a 50ml glass beaker onto the balance, zeroing the balance and putting the ping-pong ball into the beaker and then taking the reading.

m = 2.78g

I then researched the following constants from *engineering toolbox (no date)*:

$C_D = 0.50$ (for a sphere)

$p = 1.2kgm^{-3} = 1200gm^{-3}$ (I have assumed pressure of 1atm and temperature of 20°C)

I researched the diameter of a ping-pong ball from a pdf from *uphysics.com (2012)*:

diameter = 4cm = 0.04m

Therefore:

$$SA = \pi r^2$$

$$SA = \pi(0.02)^2 \text{ m}$$

Calculating k using these constants:

$$k = \frac{1}{2} \times C_D \times \rho \times SA$$

$$k = \frac{1}{2} \times 0.5 \times 1200 \times (\pi \times 0.02^2)$$

$$k = \frac{3\pi}{25}$$

To theoretically predict the horizontal distance of a bouncing ball, the initial drop height and horizontal speed must be given a value (figure 2).

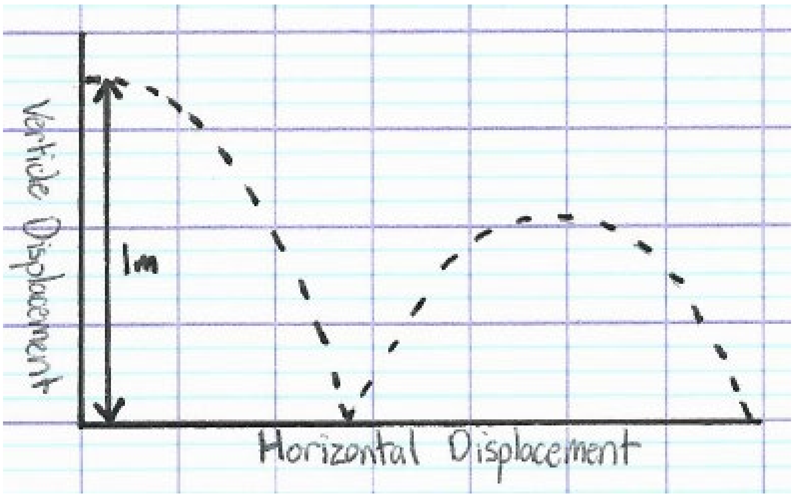


Figure 2: setting initial height (initial vertical displacement)

h = 1m

This height was a round number for me to work with and is a reasonable value in that it is not so small that the horizontal displacement and the bounce height were unreasonable.

$u_x = 0.50ms^{-1}$

To choose an initial horizontal velocity, I set up a simple experiment where I rested the ball on a stool and pushed it with my thumb to get the ball to bounce once then land in a small pot (figure 3). This took several attempts. I then imported the video into LoggerPro, after slowing the video to 0.125 normal speed, and plotted the motion of the ball every few frames. LoggerPro then calculated values for the horizontal velocity, which I divided by 0.125 to compensate for slowing the video. Using these values, I determined that $0.5ms^{-1}$ was a reasonable initial horizontal speed (also a round number which is helpful for the calculations). This test was not very reliable, however it was sufficiently accurate as a rough test to determine whether the chosen initial velocity was sensible.

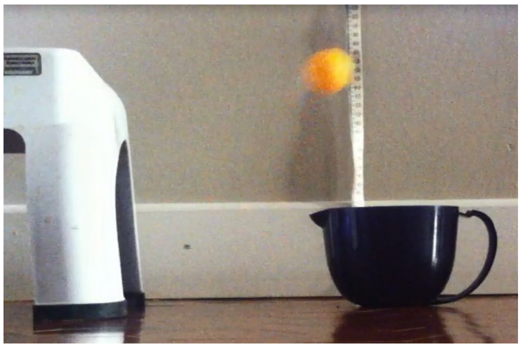


Figure 3: Rough experiment to determine initial horizontal speed

Calculation for horizontal distance of first bounce

The time of the first bounce can be calculated from the vertical component of the displacement equation (**Equation 5**) as there are only two variables, one of which, the vertical displacement, is known at the bounce. Vertical displacement is height, which, at the bounce, is zero. This then means that the time at this point can be calculated. The calculated time can then be put into the horizontal displacement equation (**Equation 6**) in order to calculate the horizontal displacement from the origin (where the ball was originally dropped.) This is possible because the time is the same for both the horizontal and vertical components, even though the two are independent.

Using **Equation 5** when s_y=0 to calculate t

$$0 = h + \frac{mgt}{k} + \frac{m^2 g}{k^2} e^{-\frac{k}{m}t} - \frac{m^2 g}{k^2}$$

I was unable to algebraically rearrange the equation to solve for 't'. Instead I substituted the constants (h, m, g, and k) into the equation which I then used to generate a graph on my calculator to solve for 't'.

Substituting constants:

$$0 = 1 + \frac{2.78 \times -9.81 \times t}{\frac{3\pi}{25}} + \frac{(2.78)^2 \times -9.81}{\left(\frac{3\pi}{25}\right)^2} e^{-\frac{3\pi}{25 \times 2.78}} - \frac{2.78^2 \times -9.81}{\left(\frac{3\pi}{25}\right)^2}$$

After plotting graph on calculator:

$$t_1 \approx 0.456s$$

$$t_1 \approx -0.447s$$

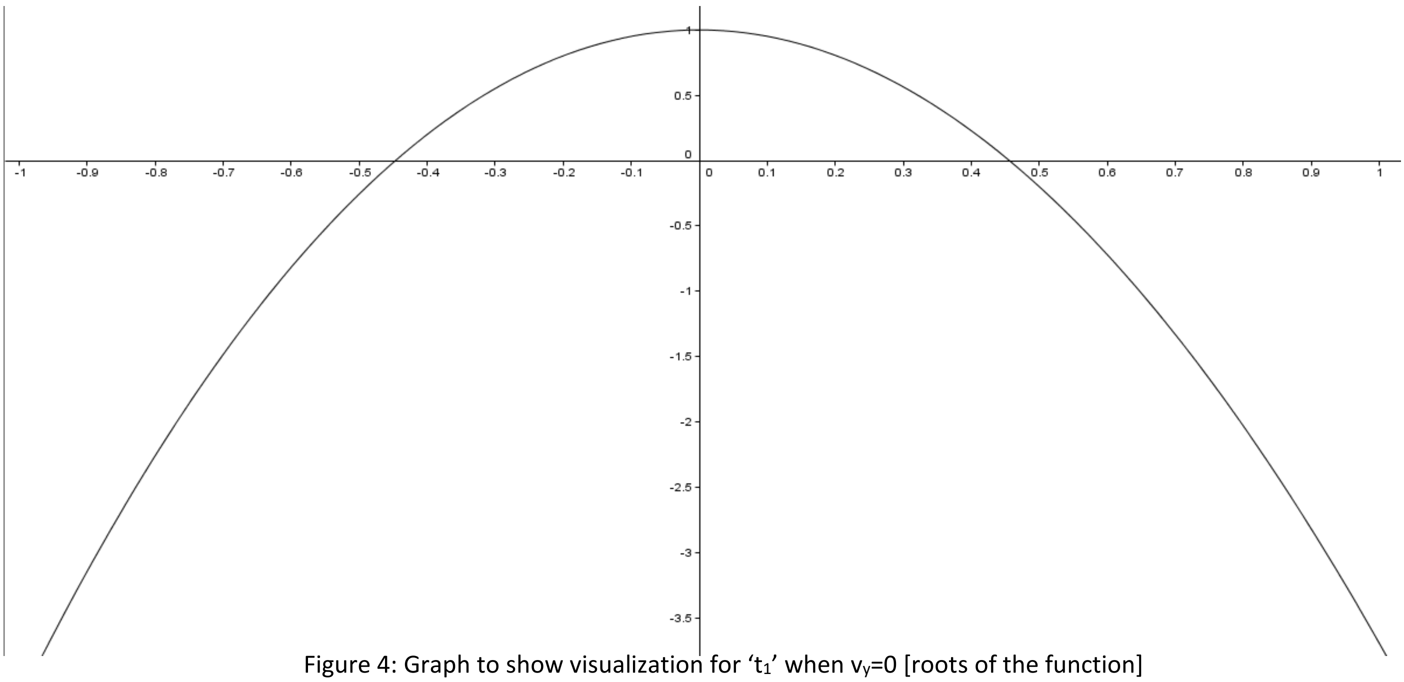


Figure 4: Graph to show visualization for 't_1' when v_y=0 [roots of the function]

The graph above (figure 4) was generated on GeoGebra as a visualization. To solve for *'t_1'* I graphed the equation on my calculator (Casio fx-CG 20) then used it to find the roots of the equation, which is when vertical displacement is 0m.

Time cannot be negative. Therefore the value of 't_1' when the vertical displacement is 0, at first bounce, is t=0.456s. To check this value, I used the SUVAT equation (International Baccalaureate, 2014) which does not take into account air resistance:

$$s = ut + \frac{1}{2}at^2$$

$$t_1 = \sqrt{\frac{1}{\frac{1}{2} \times -9.81}} \approx 0.45s$$

Therefore the value for t calculated with air resistance seems reasonable.

My first attempt to calculate the time when s_v=0m was unsuccessful. I eventually found that I had converted the radius of the ping-pong ball into cm rather than meters which changed the entire calculation. This mistake took a while to rectify as I was not sure where I had gone wrong so isolated individual factors until I found that the conversion was the problem.

Substituting value for 't_1' into **Equation 6**:

$$s_x = -\frac{mu}{k}\left(e^{-\frac{k}{m}t} - 1\right)$$

$$\therefore s_x = -\frac{2.78 \times 0.5}{\frac{3\pi}{25}} \times \left(e^{-\frac{3\pi}{25\times 2.78}\times 0.456} - 1\right)$$

$$s_x \approx 0.221m$$

Ignoring air resistance to get a rough check using a SUVAT equation (International Baccalaureate, 2014) gives:

$$s = ut + \frac{1}{2}at^2$$

As horizontal acceleration is 0

$$s = ut$$

$$s = 0.5 \times 0.45$$

$$s \approx 0.226m$$

Therefore the value for horizontal displacement calculated with air resistance seems reasonable.

Total Horizontal Displacement from drop to first bounce is $s_x \approx 0.221m$

Calculating horizontal distance for second bounce

<u>Experiment</u>

Vertically dropping a ping-ball ball allowed me to experimentally calculate the coefficient of restitution for the ball.

A meter ruler was blue-tacked to the wall while a camera, set on slow motion (8× slower than normal time), was supported by a surface and held vertically. A drop height of about 1 meter was suitable as the ball didn't have a lot of spin (so it rebounded nearly vertically) but the height, and height difference, was easily detectible on the video.

The height of the ball originally and after the first bounce can be calculated from the video. To do this, I transferred the video to LoggerPro (figure 5). Because the video was sideways, I set the origin at the base of the ruler. However this means that the x and y values in reality would actually be the opposite on the video and the values from the video will also be negative as the ball drop is to the left of the set origin, which must be taken into account when evaluating the graph. I plotted a point every few frames and stopped when the ball reached its maximum height after bounce (including a plot at that point), which could be seen visually between frames.

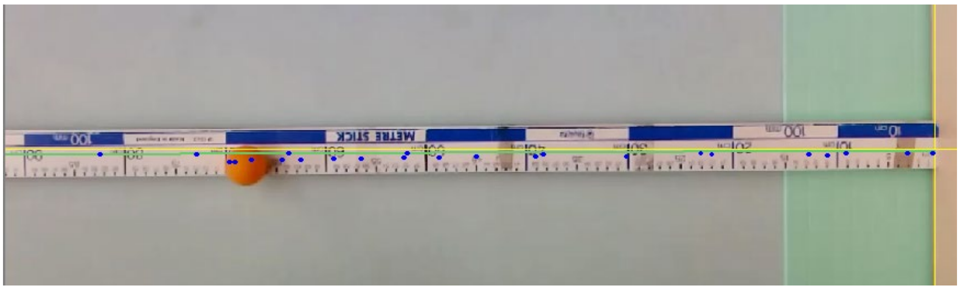
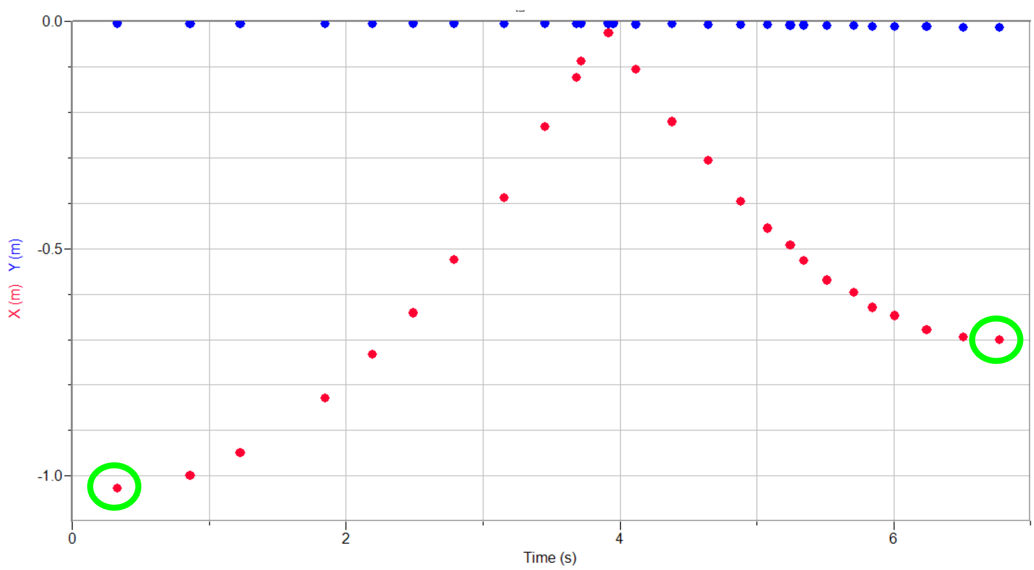


Figure 5: vertical drop height video in LoggerPro

The drop height and the maximum height after the bounce can then be determined by looking for the maximum x displacement in the table generated by LoggerPro. It can also be seen in the graph below (figure 6). The max heights are circles in green. Determining the heights in this way is a source of error, as plotting the points by hand on LoggerPro leads to inaccuracies. However it is sufficient for collecting the data required to calculate a ratio between the square of the two heights (coefficient of restitution).



Time (s)	X (m)	Y (m)	X
0.3320	-1.027	-0.004357	
0.8633	-0.9989	-0.005447	
1.229	-0.9499	-0.005447	
1.848	-0.8290	-0.005447	
2.191	-0.7331	-0.005447	
2.490	-0.6416	-0.004357	
2.789	-0.5240	-0.004357	
3.154	-0.3889	-0.005447	
3.453	-0.2320	-0.004357	
3.686	-0.1242	-0.005447	
3.719	-0.08715	-0.004357	
3.918	-0.02614	-0.004357	
3.951	-0.001089	-0.004357	
4.117	-0.1057	-0.006536	
4.383	-0.2211	-0.005447	
4.648	-0.3061	-0.007625	
4.881	-0.3965	-0.007625	
5.080	-0.4553	-0.007625	
5.246	-0.4924	-0.008715	
5.346	-0.5272	-0.008715	
5.512	-0.5697	-0.009804	
5.711	-0.5969	-0.009804	
5.844	-0.6296	-0.01089	
6.010	-0.6481	-0.01089	
6.242	-0.6786	-0.01089	
6.508	-0.6950	-0.01307	
6.775	-0.7004	-0.01307	

Figure 6: Graph and table to show the drop height and maximum height of bounce

<u>Coefficient of Restitution</u> – for ping-pong ball

The coefficient of restitution (e) is a number between 0 and 1 that is a measure of the elasticity of a collision. It is a ratio between the relative speed of approach compared to the relative speed of separation of two bodies. Therefore:

$$e = \frac{speed\ of\ separation}{speed\ of\ approach} = \frac{v_s}{v_a}$$

In an inelastic collision, some of the KE of the ball is transferred into other forms of energy and therefore the KE of the ball after the bounce is less than before the bounce. (*racquetresearch.com, no date*) This means that the height of the bounce will be less than the original drop height (figure 7).

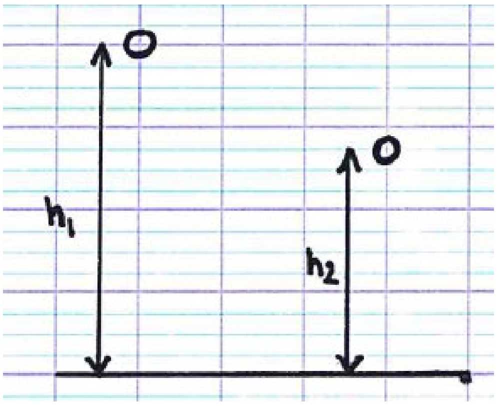


Figure 7: Illustration of max height of ball

As it is difficult to measure velocity before and after the bounce, I decided to use the heights to calculate the coefficient of restitution.

From the LoggerPro table:

h_1=1.027

h_2=0.700

Speed of approach:

$$PE_1 = mgh = 9.81mh_1$$

$$KE_1 = \frac{1}{2}mv^2$$

Assuming no air resistance

$$PE_1 = KE_1$$

$$9.81mh_1 = \frac{1}{2}mv^2$$

$$v_a = \sqrt{2 \times 9.81 \times h_1}$$

Speed of separation:

$$PE_2 = mgh_2$$

$$KE_2 = \frac{1}{2}mv^2$$

$$\therefore v_s = \sqrt{2 \times 9.81 \times h_2}$$

Calculating Coefficient of Restitution:

$$e = \frac{v_s}{v_a}$$

$$e = \frac{\sqrt{2 \times 9.81 \times h_2}}{\sqrt{2 \times 9.81 \times h_1}}$$

$$\therefore e = \frac{\sqrt{h_2}}{\sqrt{h_1}}$$

$$e = \frac{\sqrt{0.700}}{\sqrt{1.027}}$$

$$e \approx 0.826$$

The coefficient of restitution can be used, along with the speed of approach (which can be calculated from horizontal and vertical velocities), to calculate the speed of separation.

I made the assumption of no air resistance when calculating the coefficient of restitution because I calculated the ratio using the conversion of gravitational potential energy to kinetic energy, rather than forces, which makes it difficult to take air resistance into account. Air resistance is a form of friction so there would be thermal energy transferred to the surrounding air (air particles would have greater translational energy). As a result it would not be a direct conversion from GPE to KE but GPE to KE and thermal energy transferred to the air, therefore the final KE would be less than without air resistance. The thermal energy transferred to the surrounding air could be calculated from the rise in temperature of the air. However it would be difficult for me to measure this. My decision was supported when I found air resistance only had a small effect on the ball when dropped from 1m.

Speed of Approach

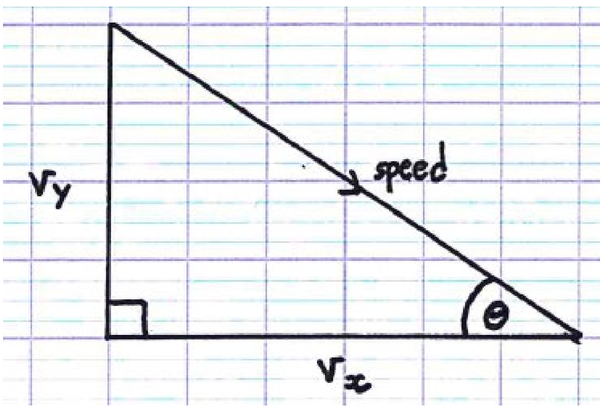


Figure 8: diagram to show relationship between speed and vertical/horizontal velocities

As the vertical and horizontal velocities have been considered independently, the speed of approach (which takes into account both velocities) can be calculated using trigonometry (figure 8).

$$speed_{apr} = \sqrt{{v_x}^2 + {v_y}^2}$$

$$speed_{apr} = \sqrt{\left(\frac{mg}{k} - \frac{mg}{k}e^{-\frac{k}{m}t}\right)^2 + \left(ue^{-\frac{k}{m}t}\right)^2}$$

$$speed_{apr} = \sqrt{\left(\frac{2.78 \times -9.81}{\frac{3\pi}{25}} - \frac{2.78 \times -9.81}{\frac{3\pi}{25}}e^{-\frac{3\pi}{25\times 2.78}\times 0.456}\right)^2 + \left(0.5 \times e^{-\frac{3\pi}{25\times 2.78}\times 0.456}\right)^2}$$

$$speed_{apr} \approx 4.365ms^{-1}$$

$$speed_{apr} \approx 4.37ms^{-1}$$

On the diagram (figure 8), the angle between $speed_{apr}$ and v_x (θ) can be calculated using trigonometry and the speed, which is known, and the horizontal velocity at the bounce, which can be calculated from the derived equation (**Equation 4**).

$$\theta = \cos^{-1}\left(\frac{v_x}{speed_{apr}}\right)$$

Where v_x is horizontal velocity:

$$v_x = ue^{-\frac{k}{m}t}$$

$$v_x = 0.5 \times e^{\frac{-3\pi}{25\times 2.78}\times 0.456}$$

$$v_x \approx 0.470$$

Therefore:

$$\theta = \cos^{-1}\left(\frac{0.470}{4.365}\right)$$

$$\theta \approx 83.82°$$

Speed of Separation

Using the coefficient of restitution:

$$e = \frac{speed\ of\ separation}{speed\ of\ approach} = \frac{v_s}{v_a}$$

$$0.826 = \frac{speed_{sep}}{4.365}$$

$$speed_{sep} \approx 3.605ms^{-1}$$

$$speed_{sep} \approx 3.61ms^{-1}$$

Horizontal and vertical velocities of separation

I am assuming the ping-pong ball has no spin, no friction and does not deform during the impact with the floor. Therefore the angle that the ball approaches the impact will be the same as the angle that the ball separates after impact (figure 9).

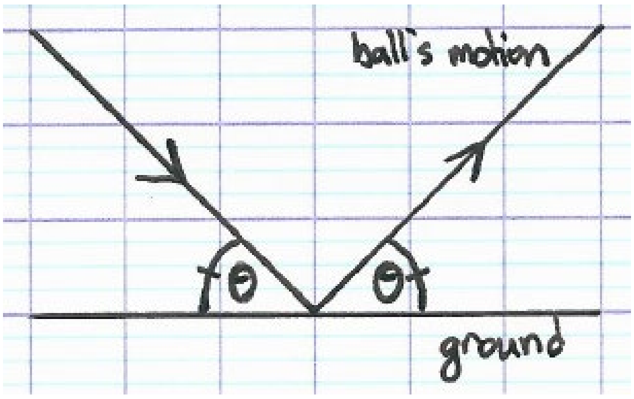


Figure 9: Angle of approach equals angle of separation

The angle, as well as the speed, are known therefore it is possible to calculate the horizontal and vertical velocities using trigonometry.

Vertical velocity:

$$\sin\theta = \frac{v_y}{speed}$$

$$v_y = \sin(83.82°) \times 3.605$$

$$v_y \approx 3.584ms^{-1}$$

Horizontal velocity:

$$\cos\theta = \frac{v_x}{speed}$$

$$v_x = \cos(83.82°) \times 3.605$$

$$v_x \approx 0.388ms^{-1}$$

Horizontal distance of second bounce

Initially I used the same equations as the ball falling to calculate time, and therefore horizontal distance. However when I tried to calculate time when the vertical velocity is zero (at the peak of the ball's motion) the calculated time was 0s. The same result was calculated when I tried to calculate the time from vertical displacement. This indicated that there was something wrong with the equation I was using. The problem, I found after checking my previous calculations for mistakes, was with the initial equation for acceleration which is not the same for a falling ball as a ball rising.

When the ball is moving upwards, the vertical net force acting on the ball will be different than the vertical net force on the ball when it is falling. Therefore the movement of the ball from the first bounce to the second bounce must be treated in two separate sections: when the ball is moving upwards and when it is falling.

Ball's motion upwards

The horizontal force of the ball does not change if it is rising or falling because the only force on the ball, air resistance, always acts against the motion; however the vertical force does change. The vertical force on the ball, unlike when the ball accelerates as the force of gravity act in the direction of the motion of the ball while air resistance opposes the force of gravity, is due to the air resistance and the force of gravity opposes the motion therefore the ball decelerates (figure 10). In addition, when the ball's motion is upwards the force of gravity is +9.81ms^{-2}.

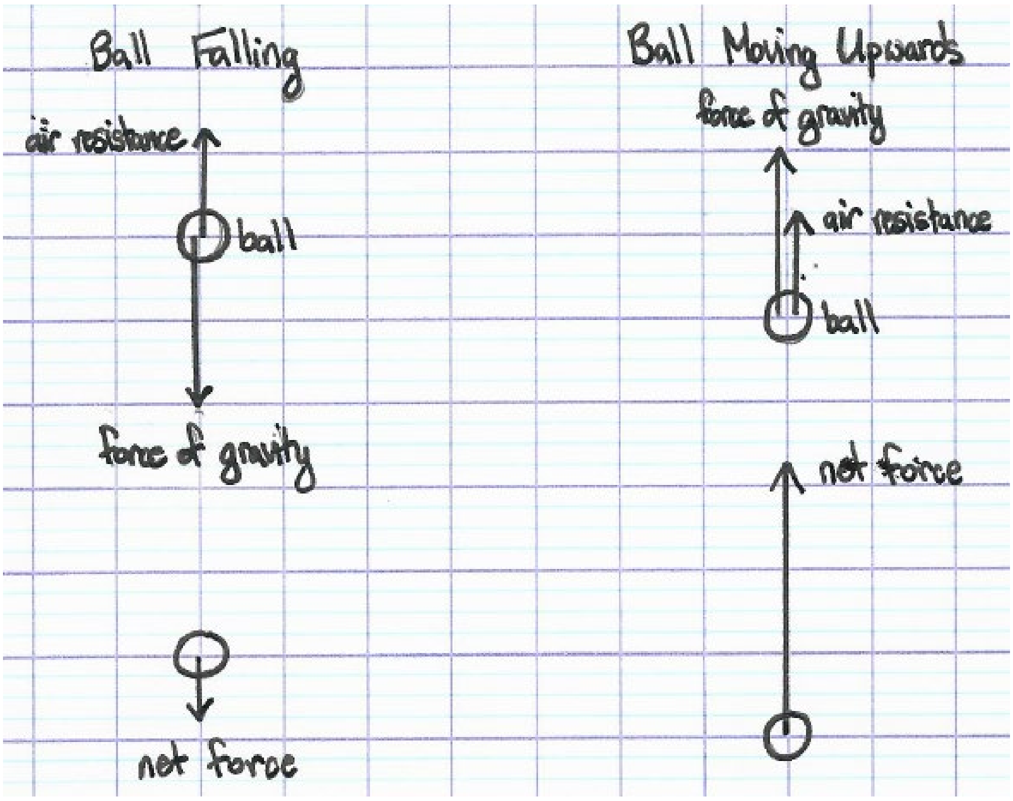


Figure 10: Free body diagram of forces on ball

<u>Solving differential equations for vertical motion</u>

$$F_{net} = -mg - kv$$

$$ma = -mg - kv$$

$$a = -g - \frac{kv}{m}$$ **Equation 7**

Since

$$a = \frac{dv}{dt}$$

This is a first order linear differential equation (to calculate velocity)

$$m\frac{dv}{dt} = -mg - kv$$

$$\frac{dv}{dt} + \frac{k}{m}t^0 v = -g$$

$$\therefore I = e^{\int \frac{k}{m}t^0 dt} = e^{\frac{k}{m}t}$$

$$e^{\frac{k}{m}t}\frac{dv}{dt} + \frac{k}{m}e^{\frac{k}{m}t}v = -e^{\frac{k}{m}t}g$$

$$\frac{d}{dt}\left(e^{\frac{k}{m}t}v\right) = -e^{\frac{k}{m}t}g$$

$$e^{\frac{k}{m}t}v = \int -ge^{\frac{k}{m}t}dt$$

$$e^{\frac{k}{m}t}v = -\frac{mg}{k}e^{\frac{k}{m}t} + c$$

$$v = \frac{-mg}{k} + ce^{\frac{-k}{m}t}$$

When t=0 then v=u_y

$$c = u_y + \frac{mg}{k}$$

Therefore

$$v = \frac{-mg}{k} + e^{\frac{-k}{m}t}\left(u_y + \frac{mg}{k}\right)$$ **Equation 8**

Integrating this gives vertical displacement

$$s = \int \left(\frac{-mg}{k} + u_y e^{-\frac{k}{m}t} + \frac{mg}{k}e^{-\frac{k}{m}g}\right)dt$$

$$s = \frac{-mgt}{k} - \frac{mu_y}{k}e^{-\frac{k}{m}t} - \frac{m^2 g}{k^2}e^{-\frac{k}{m}t} + c$$

When t=0 then s=0 (before the ball moves, vertical displacement is initial height which is 0 in this case)

$$c = \frac{mu_y}{k} + \frac{m^2 g}{k^2}$$

$$\therefore s = \frac{m}{k}\left(u_y + \frac{mg}{k} - gt - u_y e^{-\frac{k}{m}t} - \frac{mg}{k}e^{-\frac{k}{m}t}\right)$$ **Equation 9**

Calculating Horizontal Displacement

At the peak of the ball's motion, the vertical velocity will be 0. Therefore the time at the ball's peak can be calculated by putting the following equation (constants into **equation 8**) into the graphing function of the calculator to find the roots of the graph (which would be time when vertical velocity is 0):

$$v = \frac{-mg}{k} + e^{\frac{-k}{m}t}\left(u_y + \frac{mg}{k}\right)$$

$$0 = \frac{-2.78 \times 9.81}{\frac{3\pi}{25}} + e^{\frac{-3\pi}{25\times 2.78}t}\left(3.584 + \frac{2.78 \times 9.81}{\frac{3\pi}{25}}\right)$$

$$t_2 = 0.357s$$

The time can then be used to calculate horizontal displacement at the peak of the second bounce, remembering that the horizontal displacement is the same whether the ball is moving upwards or downwards, by substituting constants and the calculated 't' into **Equation 6**.

$$s_x = -\frac{mu}{k}\left(e^{-\frac{k}{m}t} - 1\right)$$

$$s_x = \frac{-2.78 \times 0.388}{\frac{3\pi}{25}}\left(e^{-\frac{3\pi}{25\times 2.78}\times 0.357} - 1\right)$$

$$s_x = 0.135m$$

In order to calculate the horizontal displacement on the ball's downward motion, the horizontal velocity and the vertical displacement at the peak (when t=0.357s) must be known. These will act as initial values for the equations of the downward motion of the ball.

Calculate the horizontal velocity by substituting constants and calculated 't_2' into **Equation 4**

$$v_x = ue^{-\frac{k}{m}t}$$

$$v_x = 0.388 \times e^{-\frac{3\pi}{25\times 2.78}0.357}$$

$$v_x = 0.370ms^{-1}$$

Calculating Vertical Displacement

Calculate vertical displacement (initial height for bounce is 0) by substituting constant and calculated 't_2' into **Equation 9**

$$s_y = \frac{m}{k}\left(u_y + \frac{mg}{k} - gt - u_y e^{-\frac{k}{m}t} - \frac{mg}{k}e^{-\frac{k}{m}t}\right)$$

$$s_y = \frac{2.78}{\frac{3\pi}{25}}\left(3.584 + \frac{2.78 \times 9.81}{\frac{3\pi}{25}} - 9.81 \times 0.357 - 3.584 \times e^{-\frac{3\pi}{25\times 2.78}\times 0.357} - \frac{2.78 \times 9.81}{\frac{3\pi}{25}} \times e^{\frac{-3\pi}{25\times 2.78}\times 0.357}\right)$$

$$s_y = 0.634m$$

Ball's Motion Downwards

With the ball's motion downwards, the vertical equations used return to the original ones. Note that the force due to gravity is -9.81ms^{-2}. The method to calculate the horizontal displacement is the same as when the ball is dropped, with different initial values.

The following equation (constants plugged into **equation 5**) is put into the graphing function of the calculator and then solved for the roots, to find time, which is when s_y=0.

$$s = h + \frac{mg}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t} - \frac{m}{k}\right)$$

$$0 = 0.634 + \frac{2.78 \times -9.81}{\frac{3\pi}{25}}\left(t + \frac{2.78}{\frac{3\pi}{25}}e^{-\frac{3\pi}{25\times 2.78}t} - \frac{2.78}{\frac{3\pi}{25}}\right)$$

$$t_3 = 0.362s$$

This time corresponds with the horizontal distance of the bounce and can be calculated by substituting constants and 't_3' into **equation 6**

$$s = -\frac{mu}{k}\left(e^{-\frac{k}{m}t} - 1\right)$$

$$s_x = \frac{-2.78 \times 0.370}{\frac{3\pi}{25}}\left(e^{-\frac{3\pi}{25\times 2.78}\times 0.362} - 1\right)$$

$$s_x = 0.131m$$

Total Horizontal Displacement from first bounce to second bounce

$$s_x = s_{x\,up} + s_{x\,down}$$

$$s_x = 0.131 + 0.135$$

$$s_x = 0.266m$$

Total Horizontal Displacement

The total horizontal displacement is the sum of the horizontal displacement from the drop to the first bounce and from the first to second bounce.

$$s_{x\,total} = s_{drop} + s_{bounce}$$

$$s_{x\,total} = 0.221 + 0.266$$

$$s_{x\,total} = 0.487m = 48.7cm$$

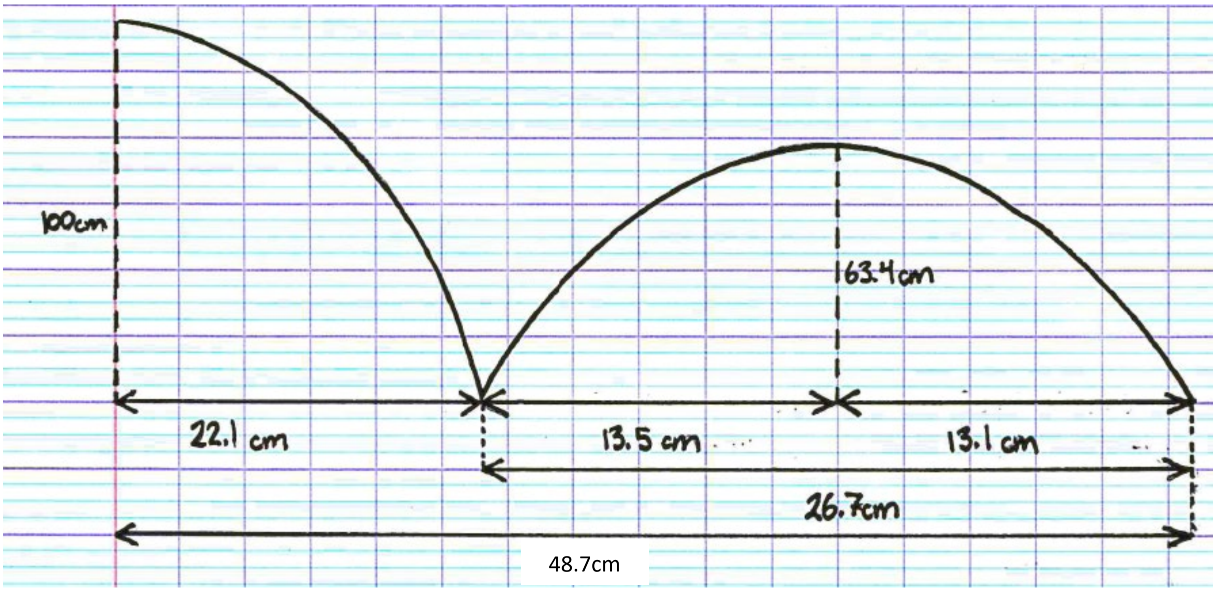


Figure 11: Summary of results

Conclusion

My aim was to theoretically calculate the horizontal distance the ping pong ball travelled from dropping to the point of impact on the second bounce. From the diagram above, the distance calculated in this investigation for a ping-pong ball dropped from a height of 1m with an initial velocity of $0.5ms^{-1}$ was 48.7cm.

Another approach I could have taken for this experiment was to start with a set horizontal distance, and an initial drop height, and attempt to work backwards to calculate the initial velocity required for the ball to travel the set horizontal distance. Alternately, I could have taken experimental data, for example videoing the motion or using a LoggerPro motion sensor, and then fitted a function to the graph I could generate from the data from the experiment.

Looking at the times of the two components from the first bounce to the second bounce, there is very little difference in the time of the ball moving upwards (0.357s) and the ball falling (0.362s). If the two times were rounded to two decimal places, they would, in fact, be the same. This suggests that air resistance did not play a significant part in the motion of the ball when dropped at 1m with an initial horizontal velocity of $0.5ms^{-1}$. While this is good in relation to the calculation of the coefficient of restitution (the ball was also dropped from about 1m) where I chose not to take air To expand the investigation, the theoretical horizontal distance with a greater initial drop height and initial horizontal velocity could be calculated, or a range of initial drop heights and horizontal velocities could be calculated, and compared to a calculation that does not take into account air resistance (SUVAT equations) in order to assess the impact of air resistance of the motion of a ball at different drop heights and initial velocities.

The coefficient of restitution would remain constant for the following bounces which suggests that the maximum height of consecutive bounces would form a sequence. This, along with a potential sequence for the horizontal distance of consecutive bounces, would be interesting to investigate further.

Returning to my assumptions, I didn't take into account any spin on the ball or friction with the surface in order to simplify the problem. This is a limitation with the model, reducing the accuracy of my conclusion. Spin may impact the way the ball interacts with the air, thereby changing its motion. However, negligible spin would have been added to the ping-pong ball if the motion had been experimentally performed so this limitation is not significant at 1m and 0.5ms^{-1}. The spin and the friction with the surface would both impact the angle at which the ball rebounded off of the surface, which would then impact the coefficient of restitution and the vertical and horizontal components of rebound speed. A final limitation with the model used is that it does not take into account the deformation of the ball at contact. These are all weaknesses in my investigation. However, I did take into account air resistance, which can be a significant factor impacting the motion of a ball. In addition, though I rounded the values in the write-up to 2 or 3 decimal places (in general), when doing the calculation I tried to use the unrounded numbers whenever possible by storing (and recalling) the numbers on my calculator.

These limitations offer an interesting extension to consider. As a volley-ball player, I have been taught to add spin to the ball so that it drops faster than it would otherwise. It would be interesting to investigate how spin, and different degrees of spin, would impact the motion of the ball. It would be difficult to measure spin, however I could potentially mark a reference point on the ball and use that to calculate angular velocity.

Besides the challenge of predicting the motion of a bouncing ball, the derived equations could be useful in areas such as computer games, such as programming a ball to bounce realistically. Though much less closely related to my experiment, using calculus, amongst other techniques, to accurately predict projectile motion is significant in areas such as launching satellites. The concept of air resistance, and calculations involving air resistance, is also important in transportation; for example when designing cars, high-speed trains or planes.

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