Finding the best Gutter shape for a House

Introduction

It is often a problem in architecture and product design, where what is mathematically optimal, is not always aesthetically what a designer wants. Compromises must be made to satisfy both parties, and new creations can be formed in the process that help push the boundaries of the architecture world. Personally, I find modern interior and exterior design fascinating, and often-times house unconventional characteristics which make them intriguing.

Figure 1 - an examples of a real rectangular gutter

[1] For example, imagine an architect has created a contemporary style house, with classic parallel and perpendicular themes as the design, to create a minimalist overview that runs inside and outside. Conventional semi-circle gutters do not fit this aesthetic. A right-angled version must be created to facilitate the designer's wishes. This must be made with the efficiency of the material

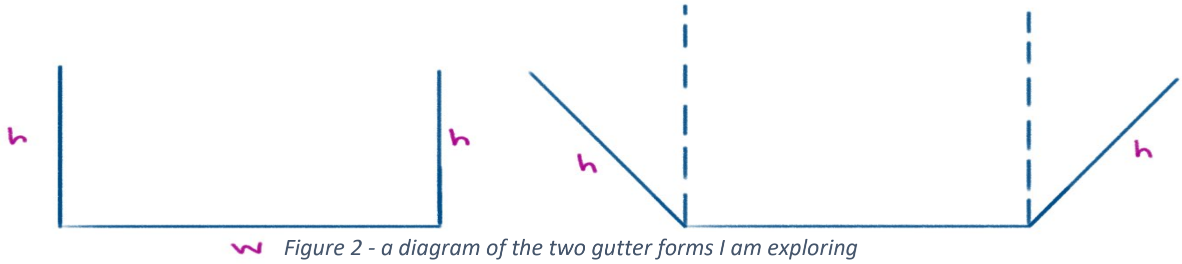

Figure 2 - a diagram of the two gutter forms I am exploring

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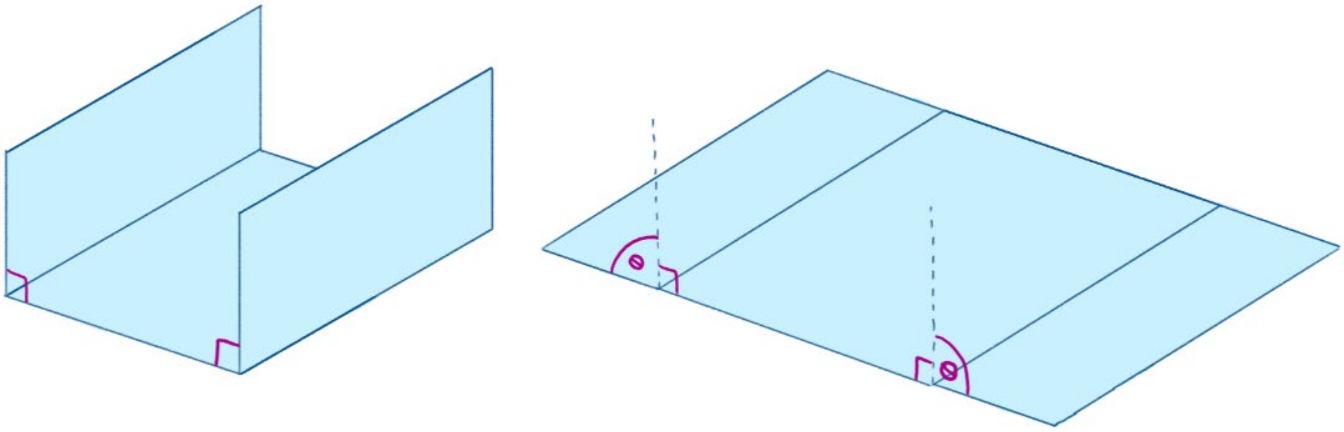
[1] *Rectangular roof gutter, Angular roof gutter - All architecture and design manufacturers* (Available at: https://www.archiexpo.com/architecture-design-manufacturer/rectangular-roof-gutter-34687.html, Last accessed 23rd January 2022)

used in mind, using optimization in mathematics. As a modern architecture enthusiast, I decided to take the task on of designing and optimizing this gutter myself.

My aim for this exploration is to deduce the optimum ratio of sides to bottom length in a set-measured, right-angled gutter. Once this is discovered and by using the same measurements, I will then form a second design for a gutter with sides that move, from a right-angled gutter to a completely flat gutter. By changing the angle of the sides and compromising the height of the gutter, the cross-section area will change too. Hence, I can find an optimum angle for the biggest area created. I will consider a rectangular cross section and see if changing the angle of the sides improves the cross-sectional area, therefore helping to maximize water flow.

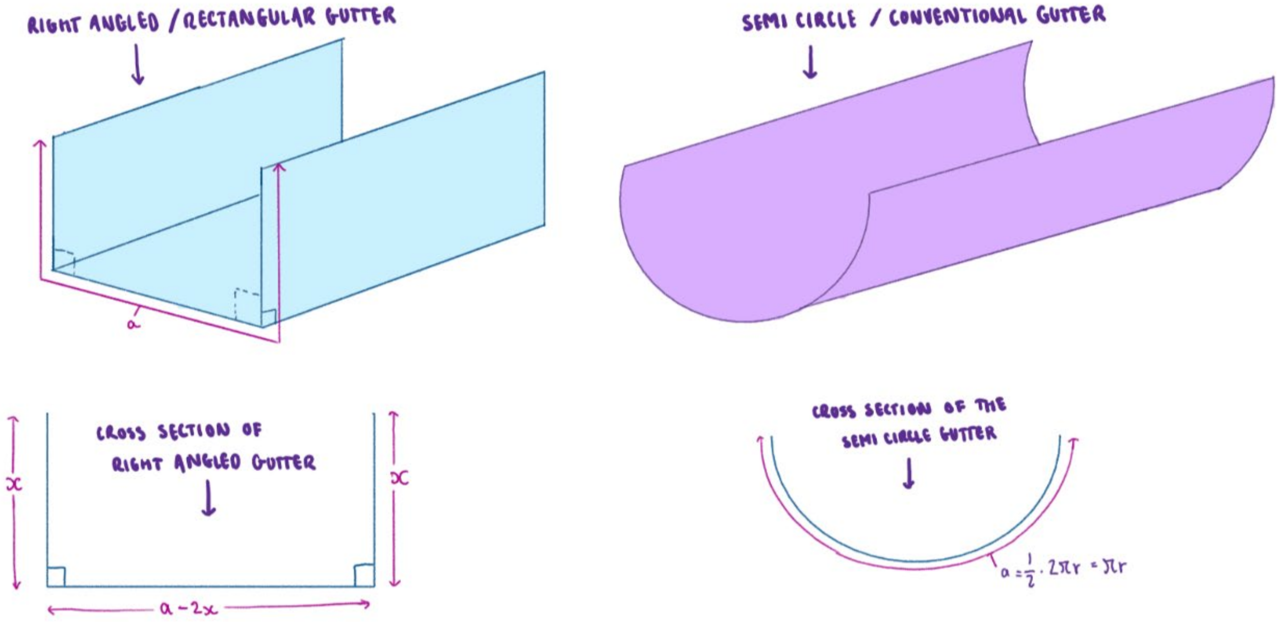
Figure 1 - Diagrams of a gutter with sloping/changing angle sides



Exploration:

To visualize the gutter’s measurements, I first drew diagrams and cross-sections of the two gutters. Here is the representation of a conventional gutter compared to a proposed right-angled gutter. The length of the three sides together is (a), the length of the two equal sides is (x), and the length of the bottom is (a-2x).

Figure 2 - Diagrams of a rectangular gutter and a conventional gutter



Now I will find the ratio of the length of the sides to the length of the bottom with an aim to maximise the area of the cross-section, while keeping the angle of the sides 90°. I will first find the sides in terms of (a).

$$Area = x\,(a - 2x)$$

$$= ax - 2x^2$$

I will differentiate to find the gradient function of the cross-section area. Next, I will set it equal to zero to find x.

$$\frac{dA}{dx} = a - 4x$$

$$0 = a - 4x$$

$$4x = a$$

$$x = \frac{a}{4}$$

Now I have the sides x in terms of a, I can find the bottom length also in terms of a.

$$The\ two\ sides\ are\ x = \frac{a}{4}\ \ and\ the\ bottom\ length\ is:$$

$$a - 2\left[\frac{a}{4}\right] = \frac{a}{2}$$

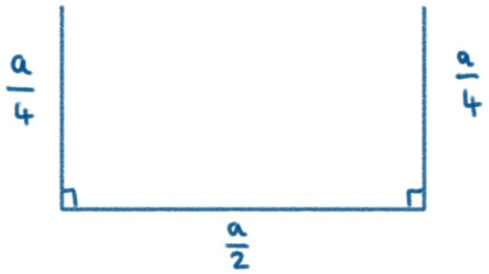


Figure 3 - Diagram to show the ratio of the sides in an optimum rectangular gutter

Now the ratio of the sides to the bottom length is found, I can substitute any value of a into it. For example, if $a = 50\ cm$, the sides will be 12.5 cm each and the bottom will be 25 cm. The ratio of the sides is 1:2, height: width.

The area equation, $A = ax - 2x^2$ can be represented as a graph. The roots can be found by making the equation = 0.

$$y = ax - 2x^2$$

$$0 = ax - 2x^2$$

$$0 = x\ (a - 2x)$$

$$x = 0,\ \ a - 2x = 0$$

$$a = 2x$$

$$x = \frac{a}{2}$$

Here is a graph to show the function of the cross-section area.

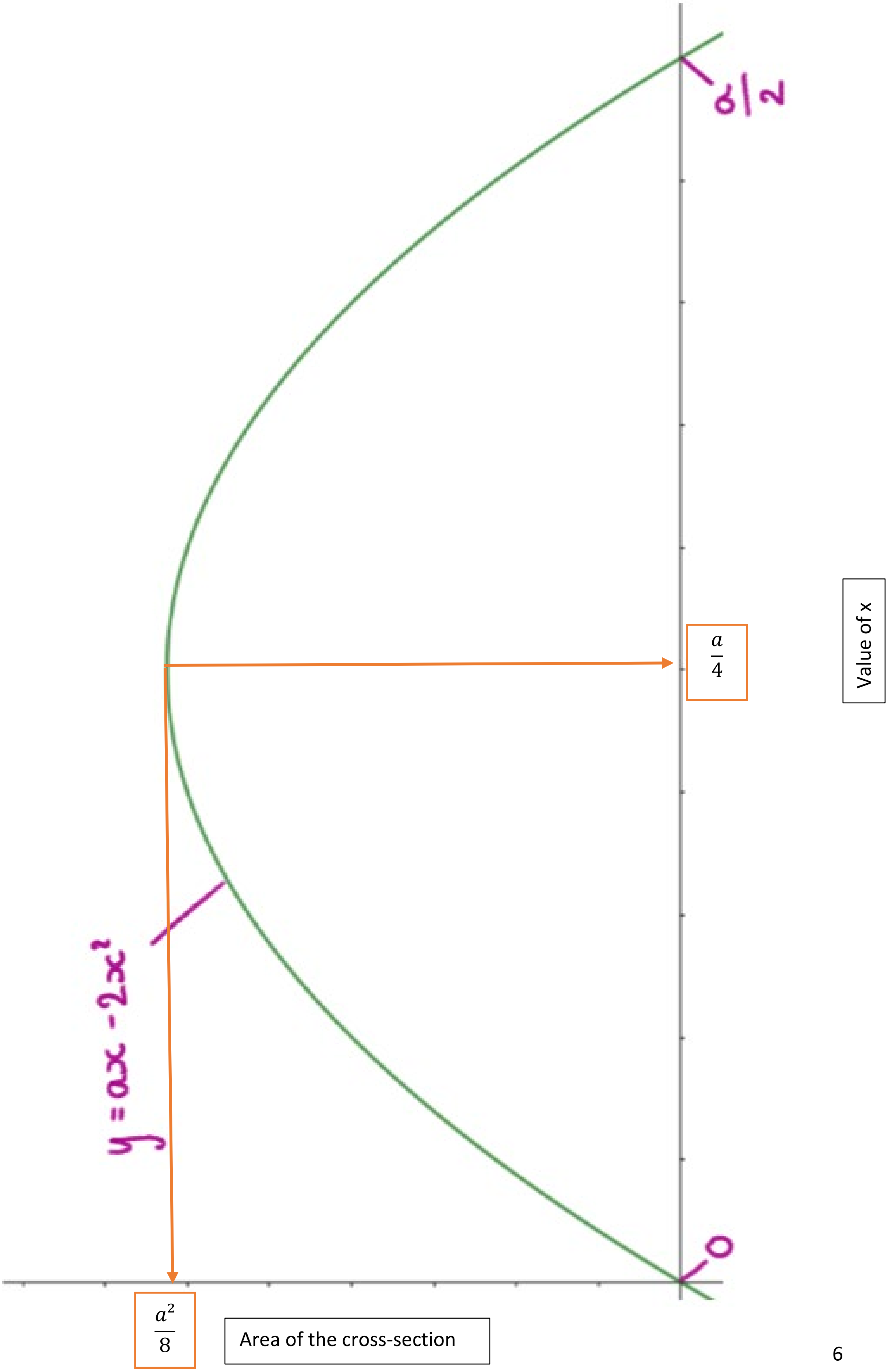
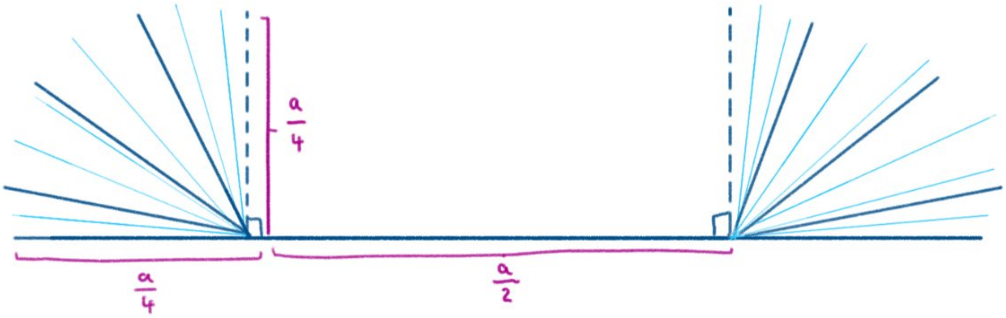


Figure 4 - A graph to show the area function of a rectangular gutter

Finding the optimum angle for a right-angled gutter with sides that change angle (0-90°)

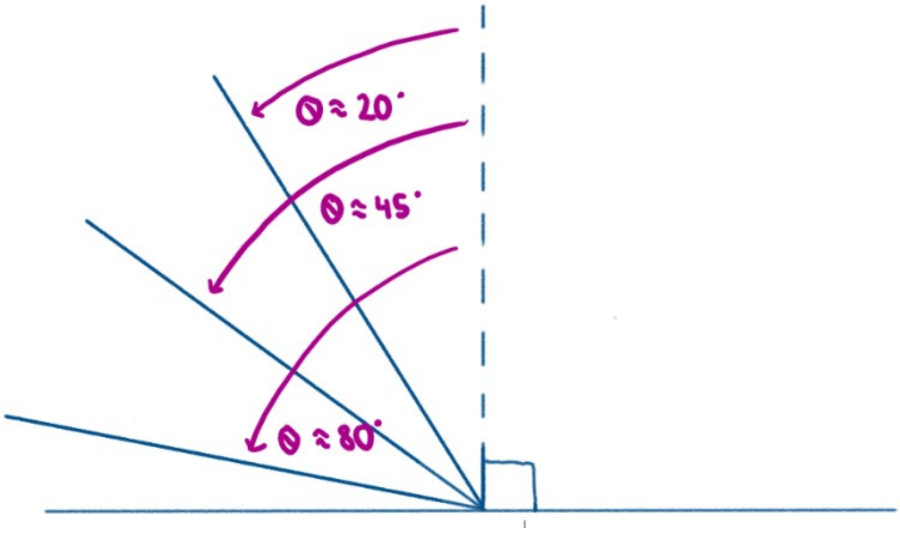
Imagine now that the sides of the gutter were able to move up and down, from all the way upright, to all the way flat, the area of the cross-section would no longer be constant. For this part of the exploration I will keep the same dimensions/ratio found in the first section.

Figure 5 - A diagram for a rectangular gutter that can change the angle of its sides



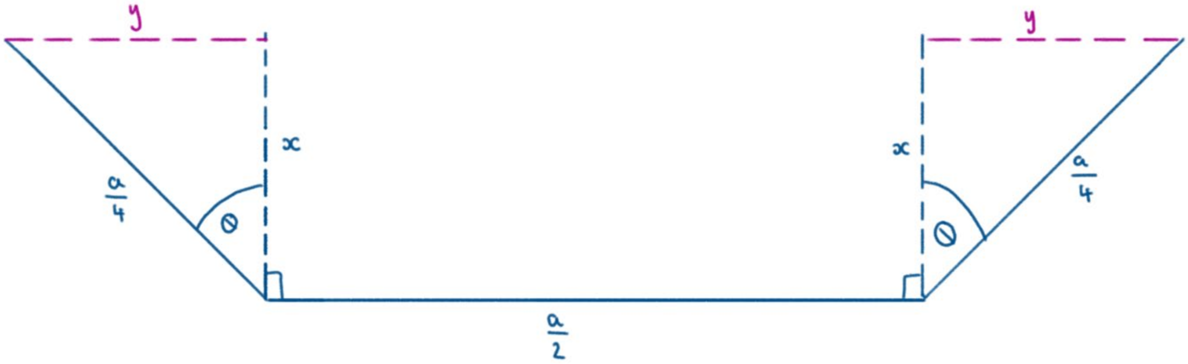
I will define θ as the angle the sides are moved by, from the 90° line. Therefore when θ = 0° the gutter is right-angled, when θ = 90° the gutter is a flat sheet.

Figure 6 - A diagram to show how the angle is measured



Firstly, I must find a function to define the area of the cross-section. The cross-section will be split up into two parts, one being the area of the triangular sections together, and the second being the area of the rectangular section.

Figure 7 - A diagram to show how the gutter can be sectioned in order to calculate its area



Working out the **triangular sections**:

$$\cos(\theta) = \frac{x}{a \div 4}$$

$$x = \frac{a}{4}\cos(\theta)$$

$$\sin(\theta) = \frac{y}{a \div 4}$$

$$y = \frac{a}{4}\sin(\theta)$$

Hence, I can find the area of the triangles together in terms of the angle.

$$Area\ of\ triangles\ together = 2 \times \frac{1}{2} \times \left[\frac{a}{4}\sin(\theta) \times \frac{a}{4}\cos(\theta)\right]$$

$$= \frac{a}{4}\sin(\theta) \times \frac{a}{4}\cos(\theta)$$

The next step to find the cross-section area is to find the area of the **rectangle section.**

$$Area\ of\ rectangle\ section = bottom\ length \times side\ length\ (x)$$

$$= \frac{a}{2}\left[\frac{a}{4}\cos(\theta)\right]$$

Now I have the area of the triangles and the rectangle section, I can add them together to get the total area of the cross-section in terms of the angle. I now have an area function which I can manipulate to get an optimum angle.

$$Total\ area\ of\ the\ cross\ section\ (A) = \left[\frac{a^2}{8}\cos(\theta)\right] + \left[\frac{a^2}{16}\cos(\theta)\sin\,(\theta)\right]$$

I will first differentiate the function, with the use of the product rule for the second bracketed section.

$$A = \frac{a^2}{8}\cos(\theta) + \left[\frac{a^2}{16}\cos(\theta)\sin(\theta)\right]$$

$$u = \cos(\theta) \qquad v = \sin(\theta)$$

$$\frac{du}{d\theta} = -\sin(\theta) \quad \frac{dv}{d\theta} = \cos(\theta)$$

$$\frac{d}{d\theta}\left[\frac{a^2}{16}\cos(\theta)\sin(\theta)\right] = \frac{a^2}{16}[cos^2(\theta) - sin^2(\theta)]$$

$$Therefore\ \frac{dA}{d\theta} = -\frac{a^2}{8}\sin(\theta) + \frac{a^2}{16}[cos^2(\theta) - sin^2(\theta)]$$

Now that I have the gradient function of the area of the cross-section, I will find when the function equals 0 in order to find the maximum point for the original function.

Firstly, I will substitute $\cos^2(\theta)$ using a trigonometry identity.

$$cos^2\,(\theta) \equiv 1 - \ sin^2(\theta)$$

$$\frac{dA}{d\theta} = -\frac{a^2}{8}\sin(\theta) + \frac{a^2}{16}[-sin^2(\theta) + 1 - sin^2(\theta)]$$

Now I can find where the function is equal to 0. I will expand the brackets and rearrange to make the equation clearer.

$$0 = -\frac{a^2}{8}\sin(\theta) + \frac{a^2}{16}[-sin^2(\theta) + 1 - sin^2(\theta)]$$

$$0 = -\frac{a^2}{8}\sin(\theta) - \frac{a^2}{8}sin^2(\theta) + \frac{a^2}{16}$$

Now I will multiply through by 16 to eliminate the fractions.

$$0 = 2\,\mathrm{a}^2\sin(\theta) + 2a^2sin^2\,(\theta) - a^2$$

Since (a) does not affect the optimum angle, I divide through by a^2 to remove it from the equation.

$$0 = 2sin^2\,(\theta) + 2\sin(\theta) - 1$$

Next, I will substitute $\sin(\theta) = x$ to make the equation a quadratic.

$$0 = 2x^2 + 2x - 1$$

This equation cannot be factorised into brackets, so I must use the quadratic formula to solve for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{4+8}}{4}$$

$$x = \frac{-2 \pm \sqrt{12}}{4}$$

$$x = \frac{-1 \pm \sqrt{3}}{2}$$

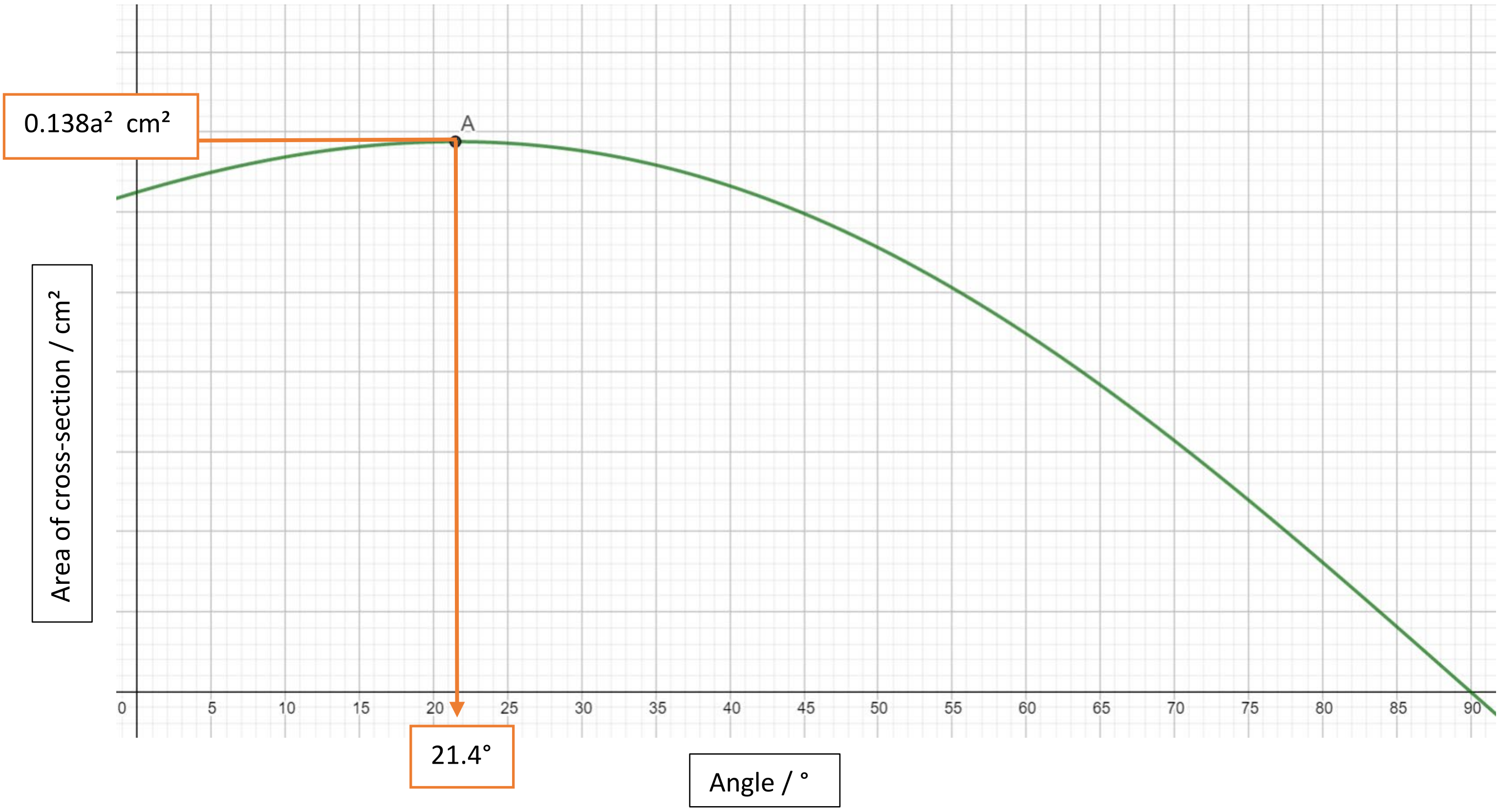
$$\sin\theta = \frac{-1 \pm \sqrt{3}}{2}$$

By using the arc sin tool on my G.D.C, I found the optimum angle. I disregarded the second value for the quadratic equation because the result was not possible, as it was a number smaller than -1.

$$\theta \approx 21.4° \; for \; 0° \leq \theta \leq 90°$$

The gradient function for the area of the cross section can be graphed. The maximum area is at $\theta \approx 21.4°$. The optimum angle is not affected by the length (a) of the cross-section, so the graph will be in terms of (a).

Figure 8 - A graph to show the area of the cross-section for a gutter with sides that change angles, for angles between 0° and 90°



The graph is shown for $0° \leq \theta \leq 90°$, the point A represents the maximum at (21.47, 0.138a²), in terms of a.

In terms of (a), the area of the cross-section for a gutter at 21.4° would be:

$$A = \frac{a^2}{8}\cos(\theta) + \left[\frac{a^2}{16}\cos(\theta)\sin(\theta)\right]$$

$$A = \frac{a^2}{8}\cos(21.4) + \left[\frac{a^2}{16}\cos(21.4)\sin(21.4)\right]$$

$$\frac{A}{a^2} = \frac{1}{8}\cos(21.4) + \left[\frac{1}{16}\cos(21.4)\sin(21.4)\right]$$

$$\frac{A}{a^2} = 0.138$$

$$\mathbf{A = 0.138a^2}\ cm^2$$

<u>Comparing the different gutter forms</u>

Now that I have the area functions for the rectangular gutter and the optimal changing angle gutter, they can be compared. I will do this by using the same value of a for each form and comparing their cross-section area.

In terms of (a):

The area of the **rectangular gutter** = $\frac{a}{4} \times \frac{a}{2} = \frac{a^2}{8}$

The area of the **angled gutter** (21.4°) = $\frac{a^2}{8}\cos(21.4) + \left[\frac{a^2}{16}\cos(21.4)\sin(21.4)\right]$

= 0.138 a^2

$$percentage\ increase = \frac{second\ value - first\ value}{first\ value} \times 100$$

$$= \frac{0.138a^2 - \frac{a^2}{8}}{\frac{a^2}{8}} \times 100$$

Now I will divide through by a^2 to simplify the equation.

$$= \frac{0.138 - \frac{1}{8}}{\frac{1}{8}} \times 100$$

$$= 0.101 \ \times 100$$

$$\approx 10.1\ \%\ increase$$

Therefore, the area of the cross-section for the changing angle gutter at optimum angle (21.4°) is a 10.1% increase on the area of the cross-section of a rectangular gutter.

An increase (and therefore improvement) in the area of the cross-section is true for only a certain range of angles which can be calculated and displayed graphically. First, I will take the area function for the rectangular gutter and find where it equals the area function for the changing angle gutter.

$$\frac{a^2}{8} = \frac{a^2}{8}\cos(\theta) + \left[\frac{a^2}{16}\cos(\theta)\sin(\theta)\right]$$

I can now divide through by a^2.

$$\frac{1}{8} = \frac{1}{8}\cos(\theta) + \frac{1}{16}\cos(\theta)\sin(\theta)$$

To simplify the equation I can multiply through by 16.

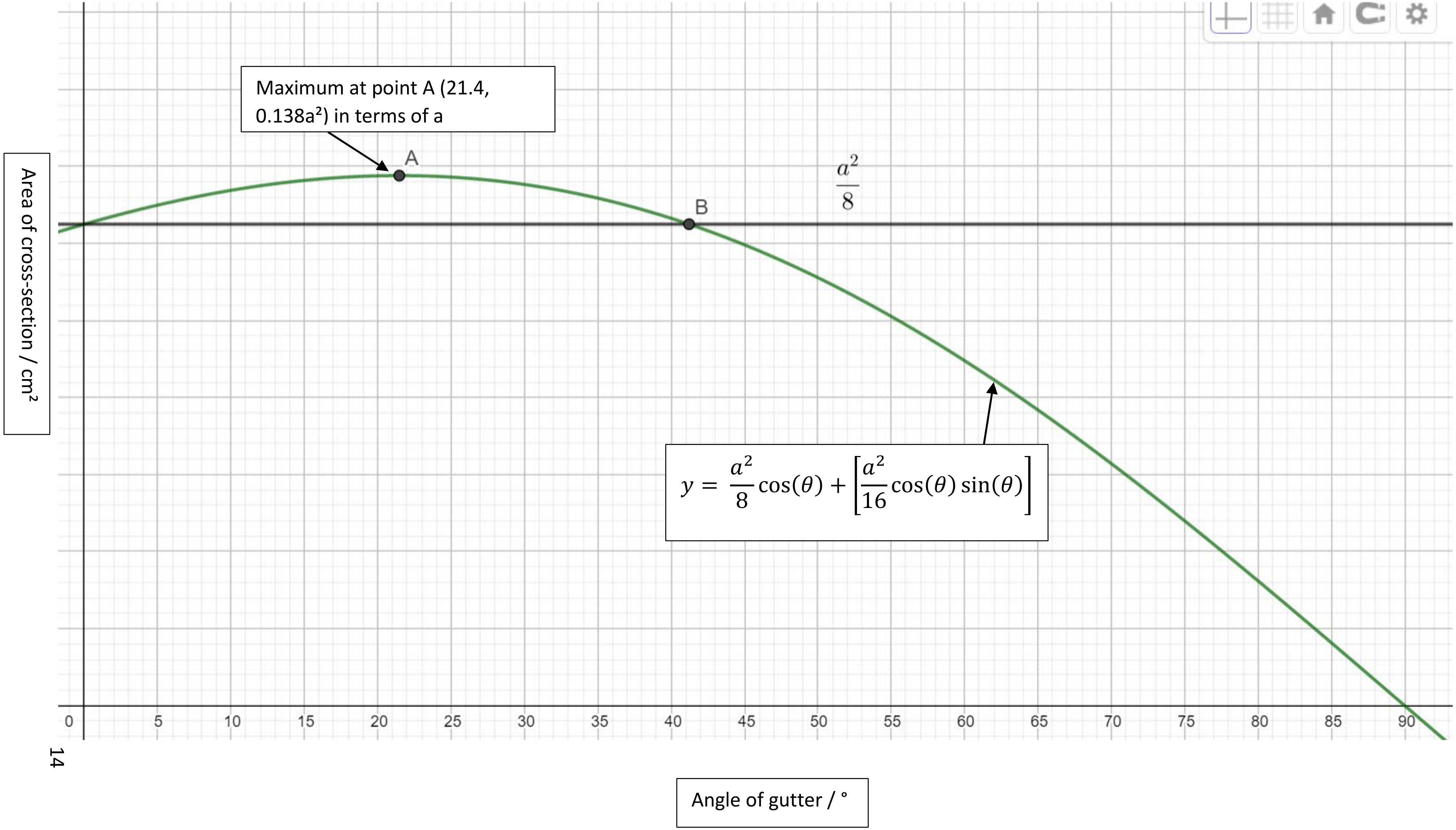
$$2 = 2\cos(\theta) + \ \cos(\theta)\sin(\theta)$$

Now I will use the G.D.C to solve this equation:

$$\theta = 0°, 41.2°$$

Therefore, the cross-section area for a changing angle gutter will be bigger than the normal rectangular gutter, only between $0° \leq \ \theta \ \leq 41.2°$.

Figure 9 - A graph to show the angles where the cross-section area of the changing-angle gutter is bigger than the normal rectangular gutter

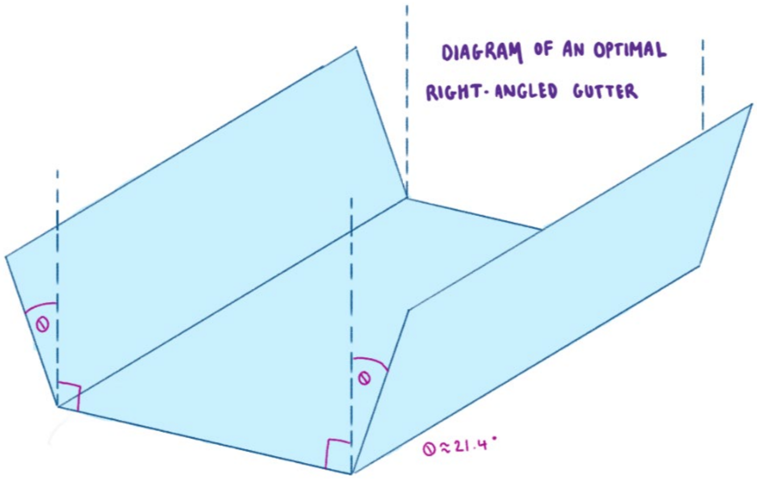


Conclusion

Ultimately, the optimum angle for the biggest cross-section area for a gutter than can change the angle of its sides is ≈ 21.4°. An increase in area and therefore increased water flow and efficacy is true for angles between 0° and ≈ 41°, beyond these values is a decrease in the area compared to a normal rectangular gutter of the same measurements (same value of a).

Through this exploration I was able to achieve my initial aims of discovering the optimisation for both a normal rectangular gutter, and for a rectangular gutter with changing angles. By exploring this piece I came across challenges I had not initially anticipated, such the complexity of the area equation for the gutter with changing angles. However, with perseverance and the use of trigonometry I was able to make the area equation and then further differentiate it. What I hadn't anticipated was being able to compare how far the changing angle gutter was an improvement on the normal one. By making the two area equations equal, I found the range of angles that improve the area and therefore water flow, and I was able to display this graphically. I used my problem-solving skills and my interest in architecture to find an answer to this intriguing maths problem.

Figure 10 - diagram of a gutter with sloping sides at an optimum angle of 21.4°



If more time were available, I would attempt to compare the two gutter forms by price to manufacture, therefore bringing a more real-life aspect to the problem. Another possible extension could be to consider adding more sides/planes to the gutter and seeing how this affects the cross-section area. They could then be compared to the rectangular or semi-circle shaped gutters. It would be interesting to investigate compromising material efficiency (semi-circle gutter) versus maximised water flow (rectangular gutter), with something in between such as a gutter with multiple planes.

Figure 11 - diagram of a gutter with two planes

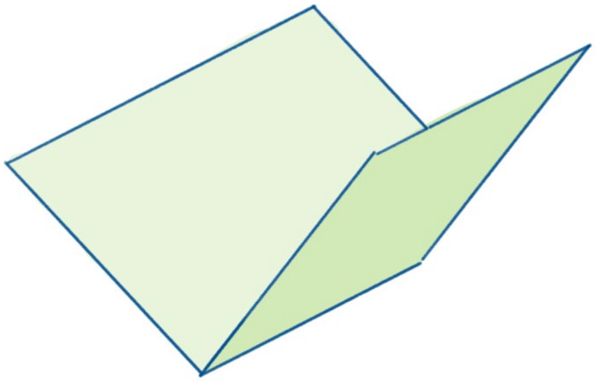


Figure 12 - diagram of a gutter with four planes

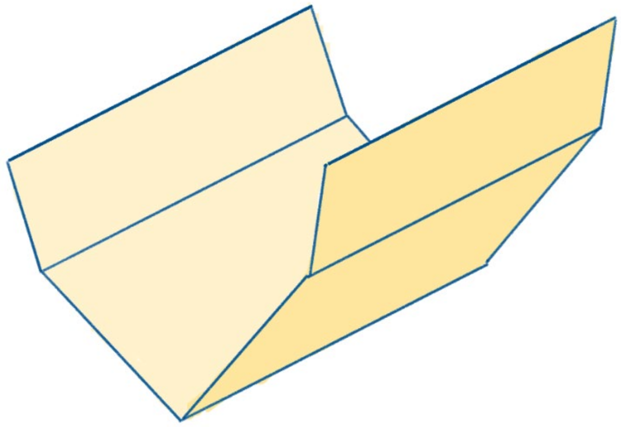
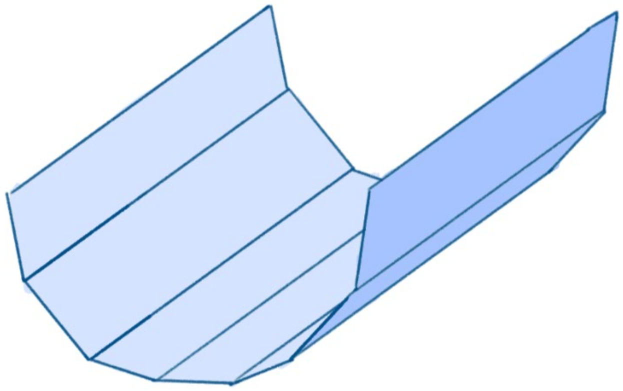


Figure 13 - diagram of a gutter with seven planes



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