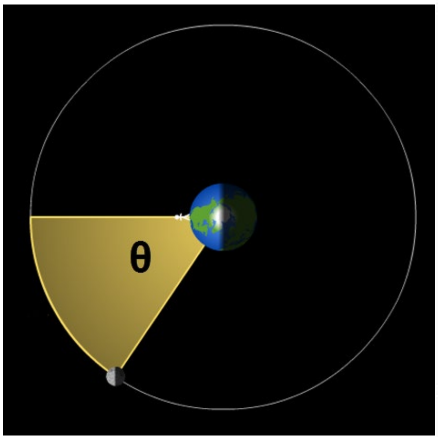
Investigation of the moon's cycles

Introduction

The moon has seven stages: new moon, waxing crescent, quarter and gibbous, full moon, and waning gibbous, quarter and crescent. These stages follow each other in a regular cycle, which is a result of the combination of the orbits of the moon and the earth. As a result, it is possible to plot the cycle of the moon on a graph. According to the United States Naval Observatory[1], their method of calculating the fraction of the moon illuminated is to calculate it for a "fictitious observer located at the centre of the Earth". The point being made by this statement is that it is essentially impossible to accurately calculate the fraction, or area, of the moon which is illuminated, as this differs widely depending not only on weather conditions, but also location and rotation of the earth, although the moon nonetheless follows regular cycles that can be tracked. It was the tracking of these cycles which allowed scientists to predict and encourage enthusiasm to the recent supermoon of November 14th[2], in which the full moon coincided with the point at which the moon is said to be in perigee: "the moon's closest point to the Earth in its monthly orbit"[3].

Choosing this topic was relatively easy for me. I've always been interested in the moon and its impact on the earth, and from a more mathematical perspective the moon's cycles seemed a perfect angle from which to explore a mathematical investigation, especially following the recent supermoon. Although there is no real use to the data that I will gain from this investigation, it is interesting to consider the values and limitations of an investigation with so many variables that cannot be controlled, as well as deriving a method to calculate area of an undefined function without using implicit integration or the equally complex professional methods used. My aim in this exploration is to find the area of the moon illuminated at a number of angles away from the sun, and then be able to find a function for the area in terms of the angle.



22.0% illuminated

[1] Astronomical Applications Department, *Fraction of the Moon Illuminated*, (Available at: http://aa.usno.navy.mil/data/docs/MoonFraction.php, Last accessed 21st February 2017)
[2] EarthSky, *what is a supermoon?* (Available at: http://earthsky.org/space/what-is-a-supermoon#what-is, Last accessed 4th March 2017)
[3] Ibid.

Data collection and processing

I used a lunar phase simulator[4] (Figure 1) which showed the orbit of the moon in relation to that of the earth, as well as the angle of the moon from sunlight, and the segment of the moon illuminated which corresponded to an angle.

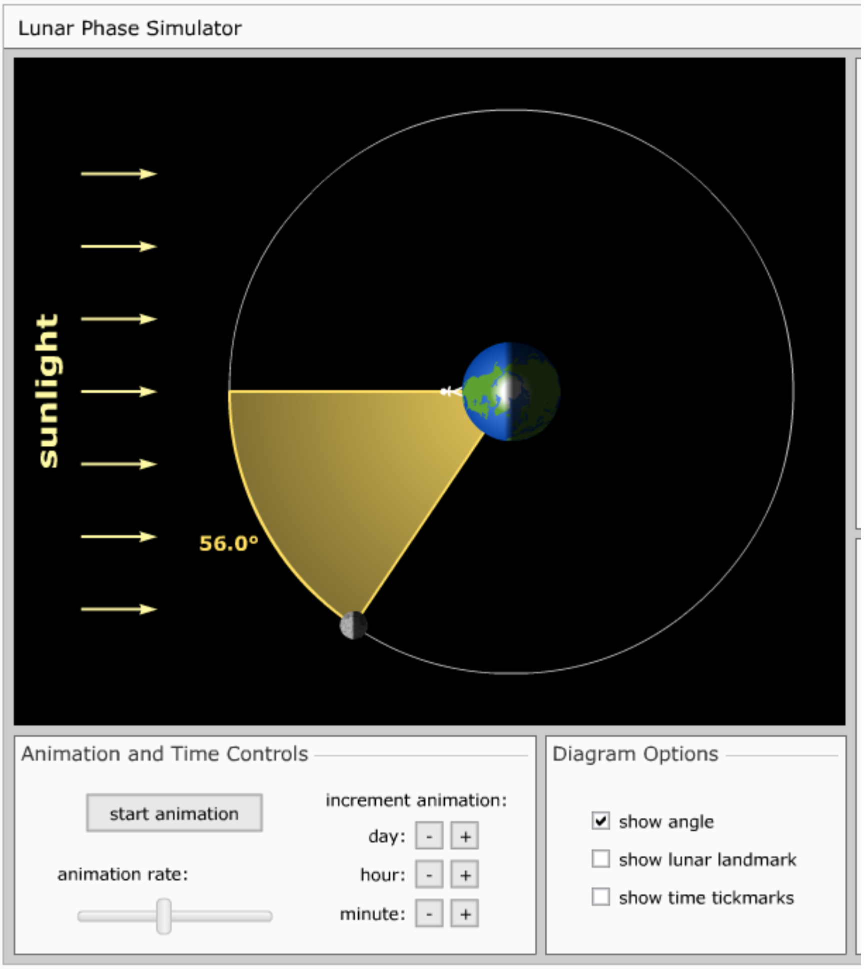


Figure 1 – Screenshot of the Lunar Phase Simulator

Initially, I imported screenshots of the "Moon Phase" section of the lunar phase simulator into Geogebra, where I would fit them to a circle with a radius of 25.0cm. I then drew a circle through three points over the illuminated segment (Figure 2).

[4] Nebraska Astronomy Applet Project, *Lunar Phase Simulator*, (Available at: http://astro.unl.edu/naap/lps/animations/lps.html, Last accessed 21st February 2017)

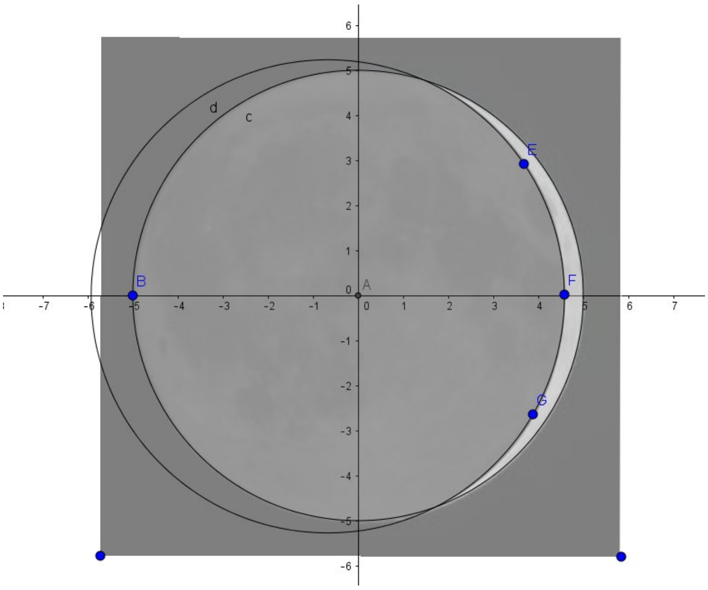


Figure 2 – Moon with illuminated segment at 25.0°

This method was not the most appropriate, as although it was effective for smaller angles, once the angle increased beyond a certain point, a slice of the illuminated segment would be lost (Figure 3). The loss of this part of the illuminated segment would mean turning already approximate calculations into wrong and inaccurate calculations, which therefore meant that I had to find a different method of calculating the area of the illuminated segment.

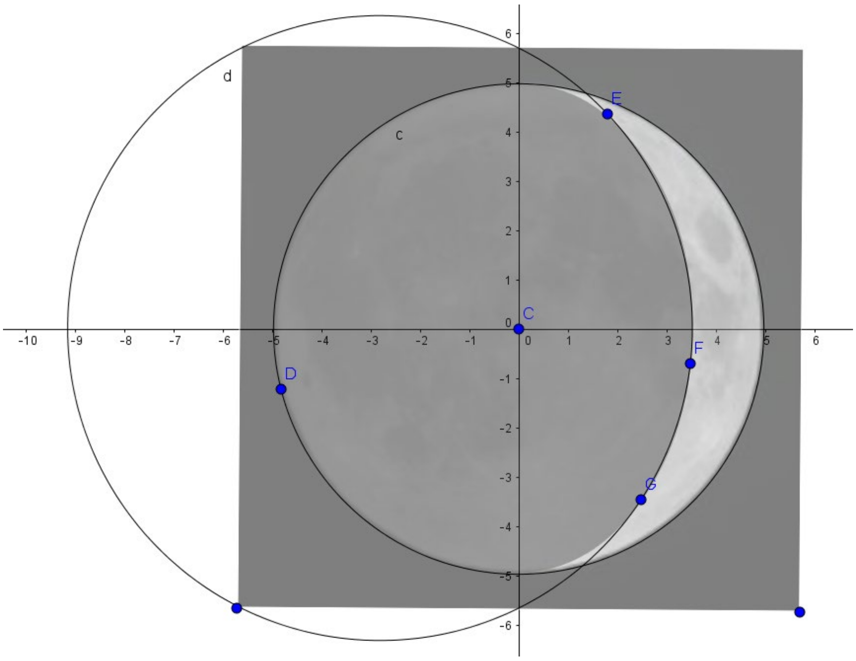


Figure 3 – Moon with illuminated segment at 45.0°

In order to find the area of the illuminated segment using this method, I would have had to use calculus in order to carry out integration of the illuminated segment. However, seeing as I didn't have functions for this graph, I would have had to either fit a function to the curve, or carry out implicit integration.

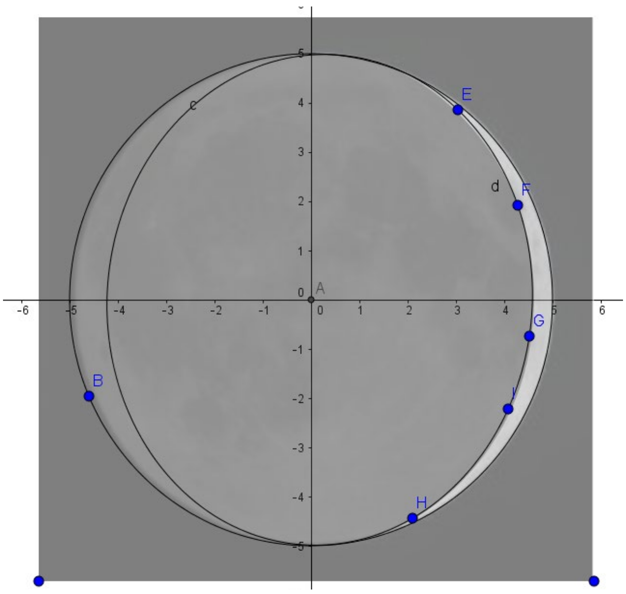


Figure 5 – Moon with illuminated segment at 25.0°

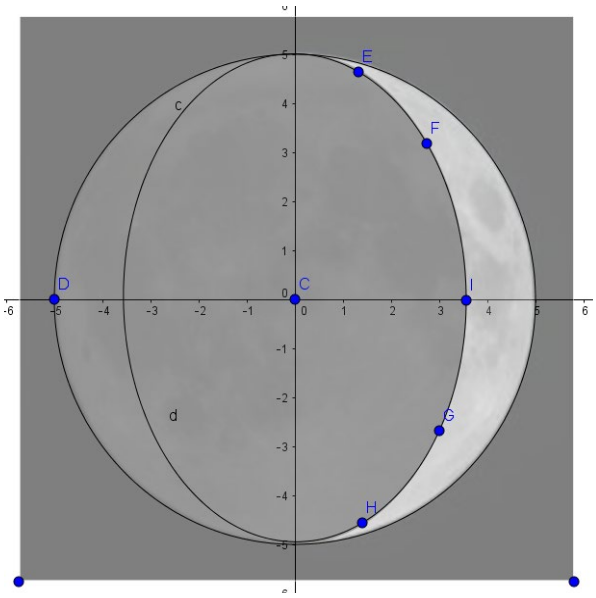


Figure 4 – Moon with illuminated segment at 45.0°

I instead chose to change my method. Still using Geogebra, I started in the same way (fitting a screenshot to a circle with a radius of 25.0cm), but instead of drawing a circle through three points, I fit a conic (ellipse) through five points. As can be seen in Figure 4 and Figure 5, which shows the same angles as above, this effectively fitted the segment illuminated much better, no matter how big or small it was. I could then proceed to the calculations.

In order to gather a range of data, I chose to use the 25.0°, 45.0°, 60.0°, 75.0° and 135.0° angles for this investigation. As the example for the calculations that I did for all of the chosen angles, I will use the 60.0° angle. The objective of the following calculations will be to find the illuminated segment of Figure 6.

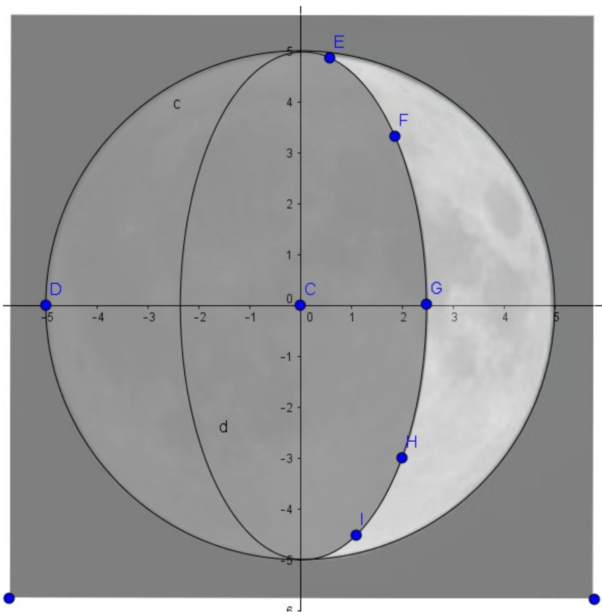


Figure 6 – Moon with illuminated segment at 60.0°

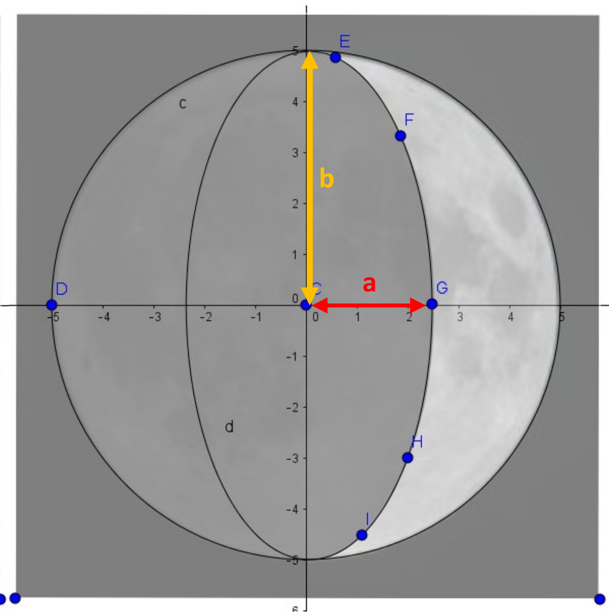


Figure 7 – Diagram showing distances "a" and "b" on the moon

The area of the circle (whole moon) is calculated with "$A = \pi r^2$."

In all instances, this was $\pi 5^2 = 25\pi$, as the moon was adjusted on the graph to fit to a radius of 5, a totally arbitrary decision as I am calculating the percentage illuminated, which would be the same regardless of radius. For future calculations, in order to only calculate the area of the illuminated segment, and not the area of the moon not covered by the ellipses, I halved this to $\frac{25}{2}\pi$.

The area of the ellipses is calculated with "$A = \pi ab$", with *"a"* being the distance from the centre to the outside of the ellipses indicated in Figure 7 and "*b*" being the radius. For the 60.0° angle circle, distance *a* was 2.48, measured with Geogebra, which made the overall calculation equal to $\frac{62}{5}\pi$. As before, for the purpose of my calculations, I halved this number to get $\frac{31}{5}\pi$.

In order to get the final area of the illuminated segment, the calculation is: "½ Area of circle – ½ area of ellipses", which in this case would be $\frac{25}{2}\pi$ - $\frac{31}{5}\pi$, which would give $\frac{63}{10}\pi$. In percentage, this would give $\frac{\frac{63}{10}\pi}{25\pi} \times 100$, or 25.2%. From a rough estimation and look at Figure 6, this seems to be an accurate number for percentage of the moon illuminated.

The overall formula for illuminated area, A= $\frac{\pi r^2}{2} - \frac{\pi ra}{2}$.

$$= \frac{\pi r}{2}(r - a)$$

The formula for the percentage illuminated is therefore: $\frac{\frac{\pi r}{2}(r-a)}{\pi r^2} \times 100$

Which can be simplified to: $\mathrm{A} = 100\frac{(r-a)}{2r}$

Which in turn is simplified to: $\mathrm{A} = 50\ \frac{(r-a)}{r}$

I used this simplified formula for the rest of the calculations done in Figure 8. Let **a** = distance from centre of the circle to outside of ellipses, θ = angle and A = percentage illuminated, as before **a** was measured on Geogebra and θ was found from the lunar phase simulator.

a	θ	A
4.58	25.0	4.2
3.56	45.0	14.4
2.48	60.0	25.2
1.32	75.0	36.8
-3.54	135.0	85.4

Figure 8 – Table showing the angle compared to the area of the illuminated segment of the moon

The simplification of the overall formula for calculating the area of the illuminated segment highlighted the importance of the value **a**, as this was essentially the factor which changed the area. Due to the connection between the angle and the area of the segment, it is possible to plot the values on a graph, which I did using Geogebra in Figure 9:

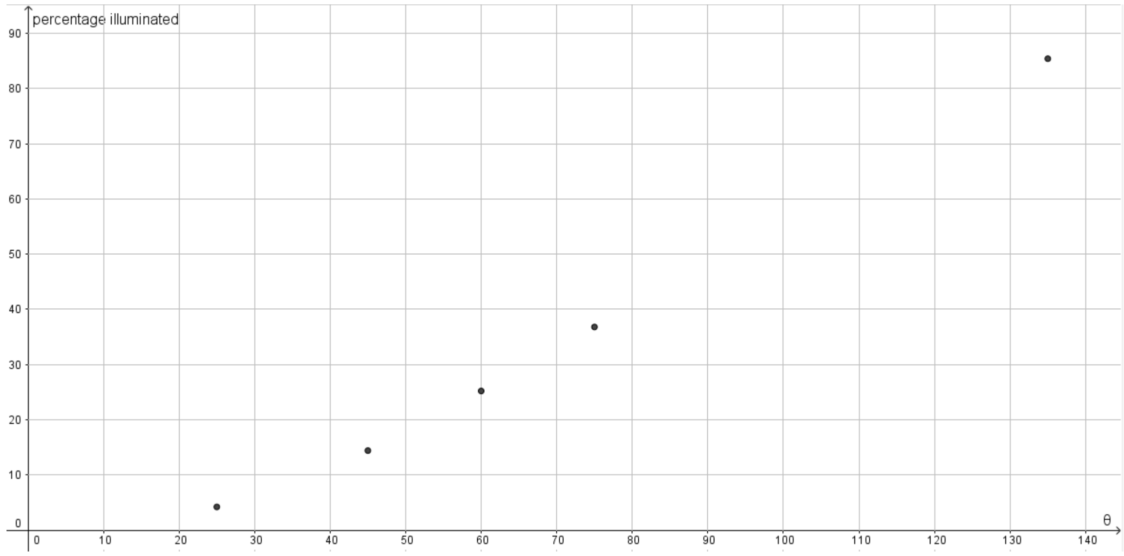


Figure 9 – Initial points plotted on a graph of percentage illuminated over ϑ

θ	percent...
10	0.6
20	3
30	6.7
40	11.8
50	17.9
60	25
70	32.8
80	41.1
90	50
100	59
110	66.8
120	74.6
130	81.6
140	87.8
150	93.4
160	96
170	98
180	100

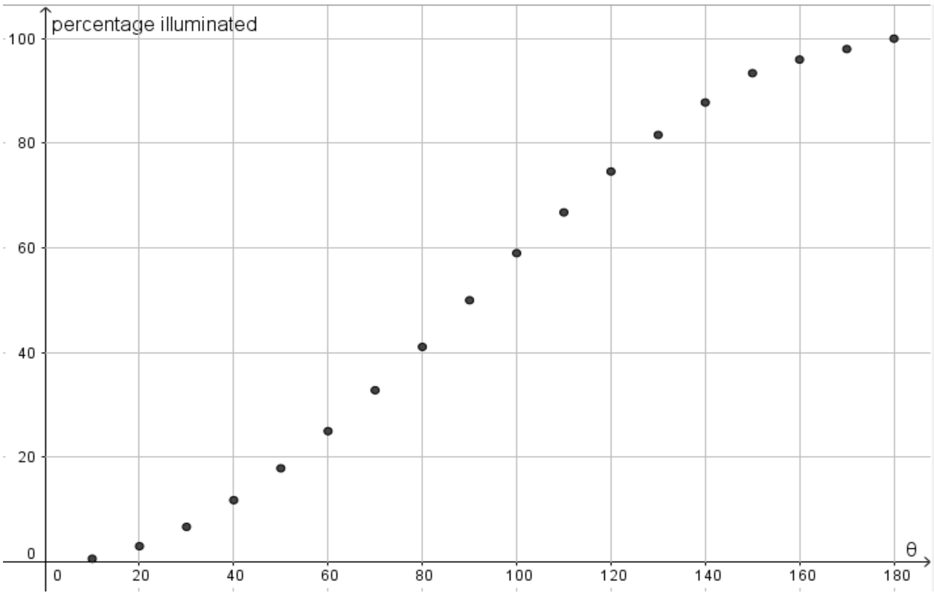


Figure 10 – points plotted on a graph of percentage illuminated over ϑ

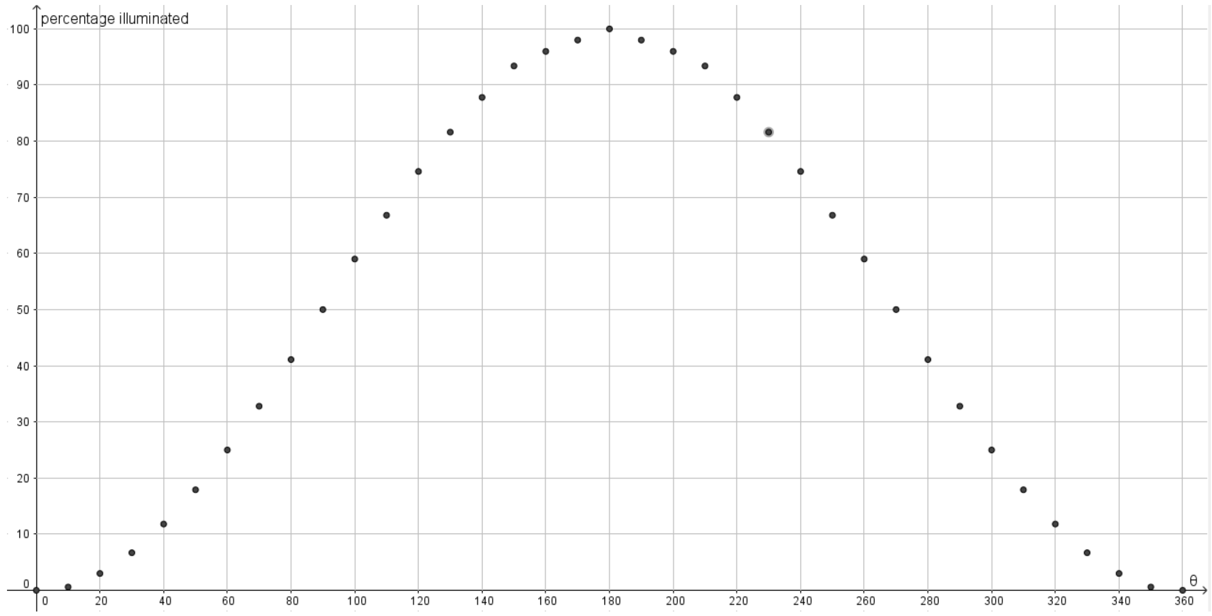


Figure 11 – full 360° cycle of points plotted on a graph

Although it looks unfinished, Figure 9 shows potential for a clearer correlation. In order to have a more complete set of results, I worked out and plotted the graph for every 10^{th} degree (0.0°, 10.0°, etc.), using the condensed method of above in order to be able to see the true correlation, which can be seen in Figure 10

From the onset, this graph looked very similar to a sine function, as Figure 11 illustrates.

In order to find the function to fit these plots, I had to calculate the amplitude, which is the maximum extent of the oscillation, the period, which is distance from one point to its successive identical point, the horizontal shift, which is movement left or right on the axes, and vertical shift, which is movement up or down on the axes.

- Amplitude (**a**) is calculated by $\frac{largest\ value - smallest\ value}{2}$, therefore $\frac{100-0}{2} = 50$
- You can calculate period (**b**) simply by looking on the graph, which would give 360: no change from the standard sine graph, and therefore no value necessary for the final equation.
- Horizontal shift (**c**) is calculated by finding the distance needed to be moved to get the minimum value to the maximum value: in this case, it would be 90.
- Vertical shift (**d**) is calculated by $\frac{largest\ value + smallest\ value}{2}$, therefore $\frac{100+0}{2} = 50$

These values give the formula as

A = 50sin (θ° – 90°) +50

where θ = angle and A = percentage illuminated

This fits near perfectly on the graph, Figure 12 .

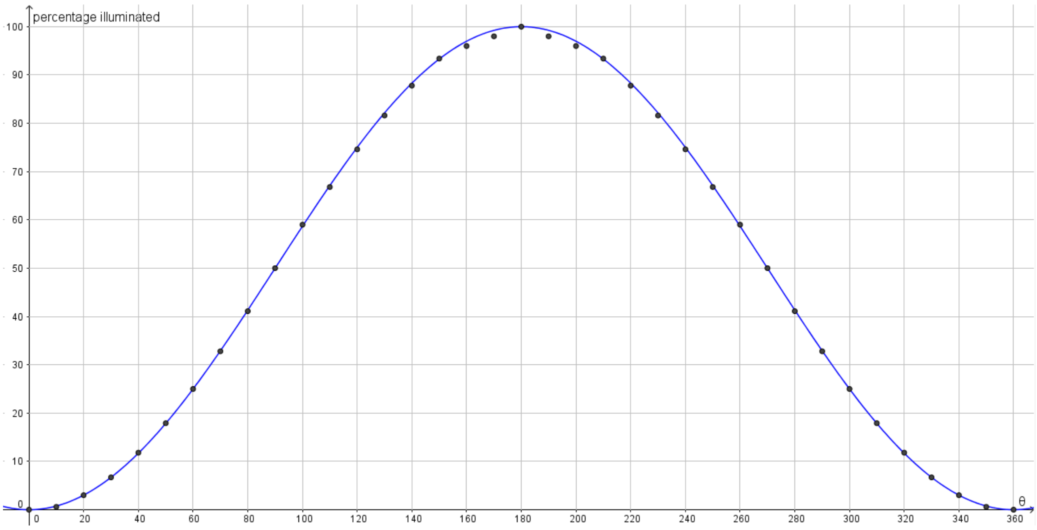
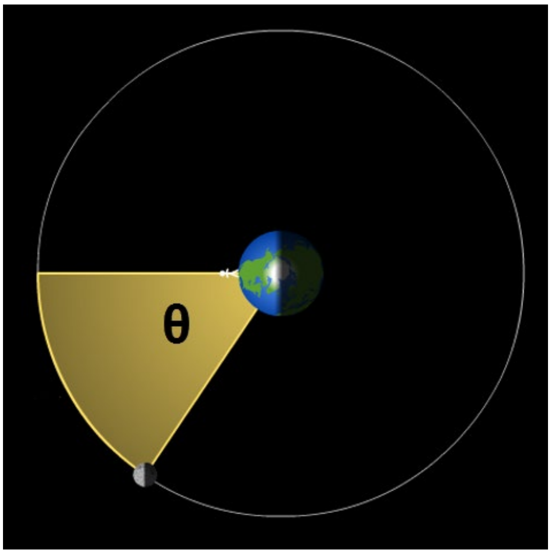
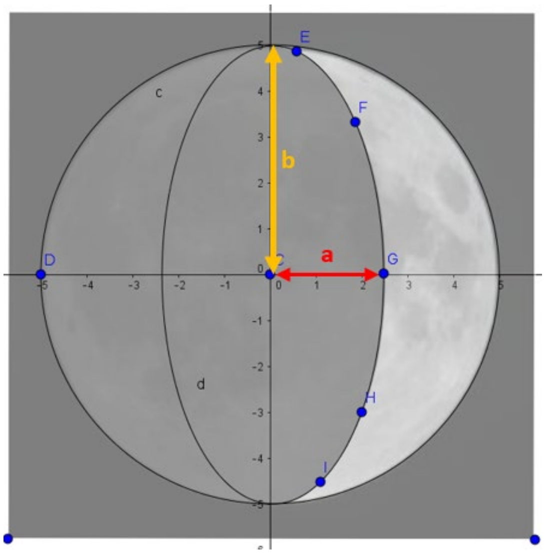


Figure 12 – graph of y= 50sin(x° - 90°) + 50 over plotted points

The only two points which diverge slightly from the function are that of (170, 98) and (190, 98). The reason for this could be an error in adjusting the ellipses to the circle or in measuring the distance ***a***, although on the whole it is a fairly unimportant divergence from the function.

At the start of this investigation, the function A= 50 $\frac{(r-a)}{r}$ was established to determine the percentage of the moon illuminated. The function above also fulfils this same purpose, and they can therefore be equated in order to give a formula for **a** in terms of θ:





50sin(**θ** ° - 90°) + 50 =50 $\frac{(r-a)}{r}$ which you divide by 50

sin (**θ** ° - 90°) + 1 =$\frac{(r-a)}{r}$ and then substituting sin **(θ-90°)=** -cos **θ**

1-cos(**θ** °) = $\frac{(r-a)}{r}$ and then multiplying by r

r - rcos(**θ** °) = $r - a$ and then subtracting r

-rcos(**θ** °) =$-a$ and divide by -1

a= rcos(**θ** °)

The equivalence of this is $a = -rsin(\theta - 90°)$. I then tested backwards in order to ascertain whether this equation is correct, using the same 60.0° value as used for an example at the beginning of the investigation. This would give $-5sin(60.0 - 90°) = 2.5$, which is the same as the value input. I tested it for the other values:

θ	Measured value for a (Geogebra)	Calculated value for a
10.0	4.94	4.92
20.0	4.70	4.70
30.0	4.33	4.33
40.0	3.82	3.83
50.0	3.21	3.21
60.0	2.50	2.50
70.0	1.72	1.71
80.0	0.89	0.87
90.0	0.50	0.00
100.0	-0.90	-0.87
110.0	-1.68	-1.71
120.0	-2.46	-2.50
130.0	-3.16	-3.21
140.0	-3.78	-3.83
150.0	-4.34	-4.33
160.0	-4.66	-4.70
170.0	-4.80	-4.92
180.0	-5.00	-5.00

Figure 13 – Graph comparing measured values to calculated values

As can be seen in Figure 13, the first measured results up to 90.0° are all very accurate compared to the calculated results. This infers very little systematic error in the method used to calculate a, and suggests that any disparities are due to random errors, such as misreading a result. The second half of the measured values show more disparity compared to the calculated values, although the same method was used to measure them. The reason that I can think of for this is that, after 90.0°, less care was taken in positioning the ellipses on the graph, which created the erroneous results. Although they aren't in accordance, they aren't far off each other either, so this isn't too major of a problem, although it is nonetheless disappointing. It does, however, show the risks of a subjective method of measurement, as well as the superior accuracy of a calculation.

<u>Assessment and conclusion</u>

There are a number of limitations to this investigation which prevent it from being wholly accurate. The first is that all of the data above relies on the information of an online applet, which means that although I've effectively graphed the moon's cycle, the data doesn't take into account the location of the individual looking at the moon or the rotation of the earth at the same time as the rotation of the earth, which would affect how an individual sees the illuminated segment of the moon. It also doesn't take into account the orbit of the moon: the applet displays the orbit as circular, whereas it is in fact elliptical. The elliptical orbit of the moon explains the occurrence of a supermoon, as logically there would be moments at which the moon is closer to the earth and would appear "bigger". In this sense, therefore, the model created by this investigation can only be used in the broadest illustrative sense.

Another limitation of this investigation is that although the calculations are, to the best of my knowledge, correct, they lack the inclusion of several variables which would make them more accurate and realistic, such as the aforementioned rotation of the earth and position of the individual, as well as season or position in the moon's orbit. The lack of these variables means that although my data is as accurate as possible with the use of an online Lunar Phase Simulator and could be used to model the technicalities of the lunar cycle, it is not applicable to real life.

Despite these limitations, the investigation remains successful, as it effectively proved that the moon follows a cycle which may be put to a trigonometric function, as well as demonstrating a correlation between the values of "*a*" and the angle of the moon away from the sun. The value "*a*" is arguably the most important value of this entire investigation, as it was this value which changed the area of the ellipses and enabled the calculation of the area of the moon illuminated.

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