# Investigation into Domes and Ellipsoids

## INTRODUCTION

I am interested in a career in architecture which led me to research the history of churches, where I came across Christopher Wren, a well renowned architect during the sixteen-hundreds who designed St. Pauls Cathedral, situated in London, Britain. He drew inspiration from both from St. Peter's Basilica and Church of the Val-de-Grâce to design the dome of the Cathedral. From a design stand point it is visually hard to see the significant difference between the three domes. This enticed me to look at the mathematics of the domes to compare their surface area and volume. The aim of my investigation is to find the most heat efficient dome, therefore the dome with the smallest surface area to volume ratio. Greater surface area causes greater heat gain and loss due to more material at the surface of the shape, which allows more energy to pass from the enclosed space to the exterior. To compare the domes, I will find a function that best fits against the image. Then I will use calculus to find both the volume and surface area. I can therefore compare the three ratios to find the dome that retains the most heat.

*Figure 2 - St. Paul's Cathedral*

*Figure 3 - St. Peter's Basilica*

*Figure 1 - Val-de-Grace*

# EXPLORATION

## Testing functions:

To find the function of each dome on GeoGebra, the inserted image of the dome must be taken parallel to the camera and the axis adjusted for the scale to fit the measured diameter (1 unit = 1 meter). St. Pauls Cathedral has an outer diameter of 34m.

Test models for St. Pauls Cathedral:

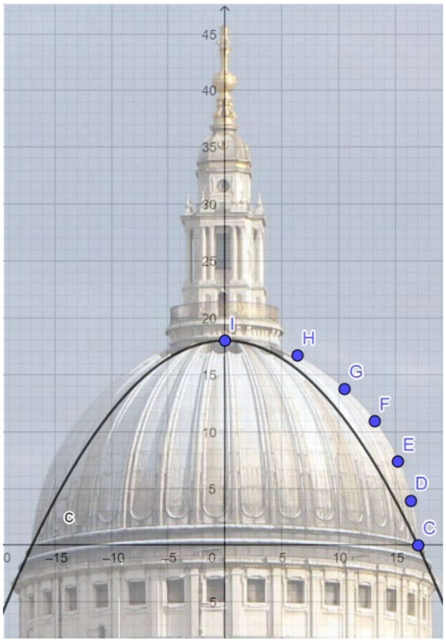



*Figure 5 - Parabolic function – poor fit*

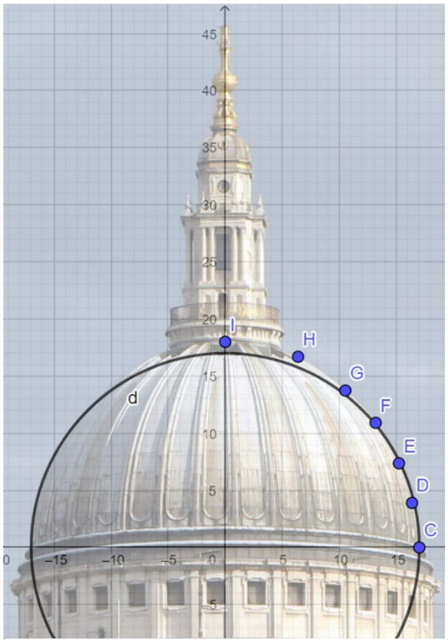



*Figure 4 – Circle model – discards top of dome*

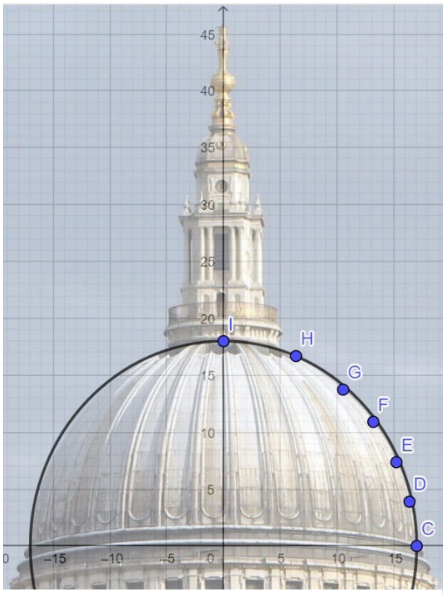


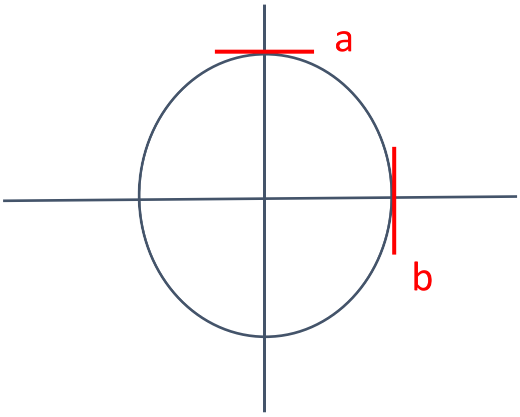

*Figure 6 - Ellipse Model - best fit*

The ellipse model is the best fit to the image. The general equation for an ellipse with its major axis oriented vertically is, $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ , with the centre at (0,0)

Therefore, the equation of the ellipse of St. Pauls Cathedral becomes, $\frac{x^2}{17^2} + \frac{y^2}{18^2} = 1$ , where $b < a$

b = 17, half the width of dome

a = 18, height of dome




St Peter's Basilica also conforms to an ellipse:

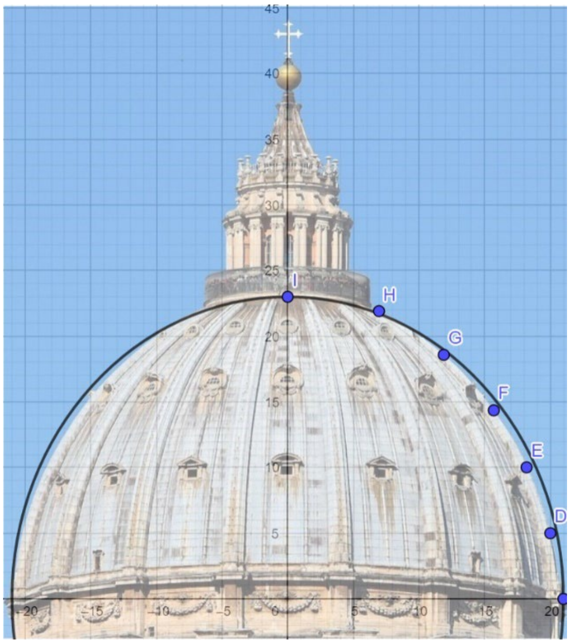



*Figure 7 - Ellipse - best fit*

St. Peter's Basilica has a diameter of 42m, therefore, the equation of its ellipse becomes,

$\frac{x^2}{21^2} + \frac{y^2}{23^2} = 1$ , with centre at (0,0)

Church of the Val-de-Grâce is rather a hemisphere as the height and radius are equal, a=b, with a diameter of 22m. This is shown below:

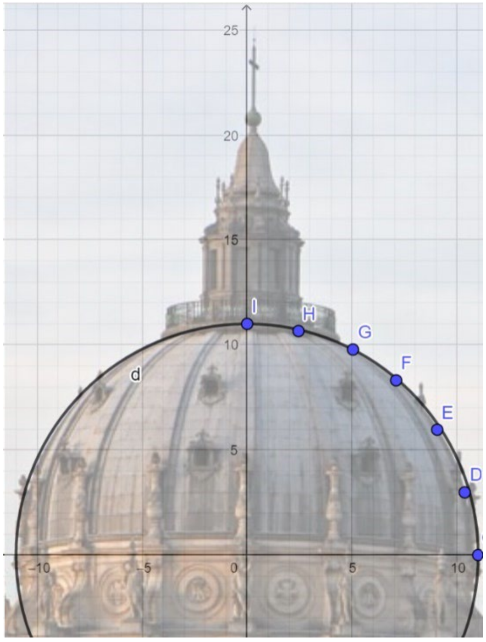


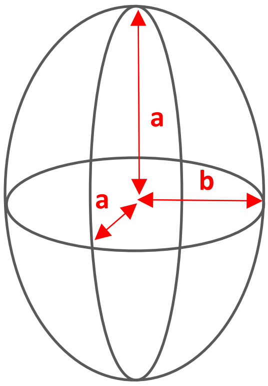

*Figure 8 - circle - best fit*

The general equation of a circle is $x^2 + y^2 = r^2$, with centre at (0,0), therefor the equation for Church of the Val-de-Grâce becomes, $x^2 + y^2 = 11^2$

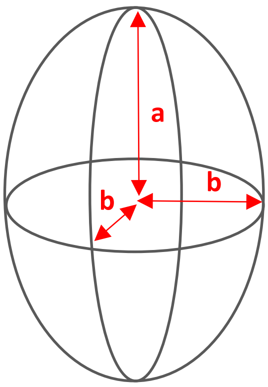
### Volume:

A different ellipsoid is created when rotated around the x and y axis;

$Around\ x\ axis, V\ = \pi\ a^2 b$




$Around\ y\ axis, V\ = \pi\ ab^2$



Therefore, to calculate the volume of St. Pauls Cathedral dome and St. Peter's Basilica dome, the ellipse must be rotated around the y axis as the minor axis is both the width and depth of the dome and the major axis is vertically positioned.

The ellipsoid equation must be rewritten in terms of $x^2$ in order to integrate the equation,

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{b^2} = 1 - \frac{y^2}{a^2}$$

$$\frac{x^2}{b^2} = \frac{a^2 - y^2}{a^2}$$

$$x^2 = \frac{b^2}{a^2}\,(a^2 - y^2)$$

With this, rearranging the volume of the revolution for St. Pauls Cathedral dome around the y axis can be found by using the disk method with its following formula:

$$V = \pi \int_0^a x^2 dy$$

*general equation*

$$V = \pi \int_0^a \frac{b^2}{a^2}\,(a^2 - y^2) dy$$

*plug values into equation*

$$V = \pi \int_0^{18} \frac{17^2}{18^2}\,(18^2 - y^2) dy$$

$$= \frac{17^2}{18^2}\pi \left[18^2 y - \frac{y^3}{3}\right]_0^{18}$$

$$= \frac{17^2}{18^2}\pi \left[18^3 - \frac{18^3}{3}\right] - [0]$$

$$V = 3468\pi \approx 11000\, m^3$$

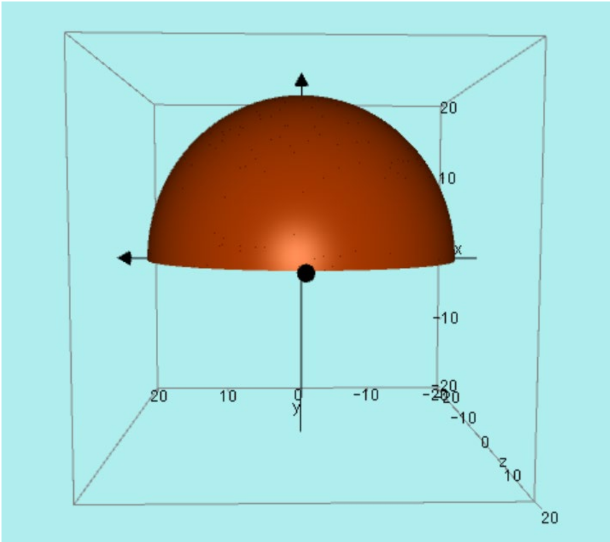



*Figure 9 – Diagram of ellipsoid*

The same method can be used to find the volume for St. Peter's Basilica dome:

$$V = \pi \int_0^a x^2 dy$$

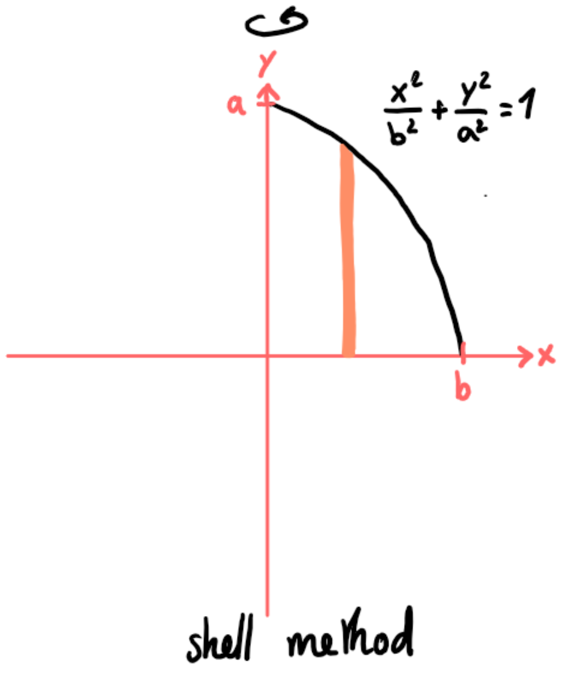
$$V = \pi \int_0^{23} \frac{21^2}{23^2} (23^2 - y^2) dy$$

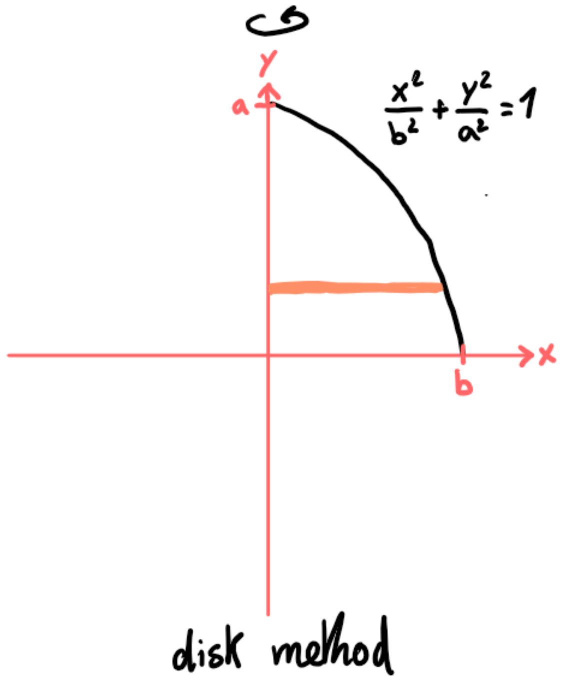
$$= \frac{21^2}{23^2} \pi \left[ 23^2 y - \frac{y^3}{3} \right]_0^{23}$$

$$= \frac{21^2}{23^2} \pi \left[ 23^3 - \frac{23^3}{3} \right] - [0]$$

$$V = 6762\pi \approx 21000\, m^3$$

This result conforms with the fact that the height, width and depth is greater than the equivalent values of St. Pauls Cathedral. The shell method is another way to solve the volume of these two semi ellipsoids which can act as a verification of the results. The difference between the two methods is that the disk method uses rectangles perpendicular to the axis of revolution, opposite to the shell method, which uses rectangles parallel to the axis of revolution, as seen in the diagrams below.




The rotation of the ellipse still occurs around the y-axis, with its following equation,

$$V = 2\pi \int_0^b r(x) \cdot h(x)\, dx$$

$$r(x), represents\ the\ distance\ from\ the\ axis\ of\ rotation$$

$$h(x), represents\ the\ height\ of\ the\ dome$$

$$V = 2\pi \int_0^b x\, h(x)\, dx$$

$$The\ ellipsoid\ equation\ is\ rearanged\ in\ terms\ of\ x\ which\ is\ h(x)$$

$$V = 2\pi \int_0^b x \left(\frac{a}{b}\,(b^2 - x^2)^{\frac{1}{2}}\right) dx$$

$$V = \frac{2a\pi}{b} \int_0^b x\,(b^2 - x^2)^{\frac{1}{2}}\, dx$$

$$Use\ intergration\ by\ recognition:$$

$$\int f'(x) \cdot \big(f(x)\big)^n\, dx = \frac{(f(x))^{n+1}}{n+1}$$

$$where, f(x) = (b^2 - x^2)$$

$$multiply\ integral\ by\ -\frac{1}{2}$$

$$V = -\frac{2a\pi}{2b}\left[\frac{2\ (b^2 - x^2)^{\frac{3}{2}}}{3}\right]_0^b$$

$$V = -\frac{a\pi}{b}\left[\frac{2\ (b^2 - b^2)^{\frac{3}{2}}}{3}\right] - \left[\frac{2\ (b^2 - 0^2)^{\frac{3}{2}}}{3}\right]$$

$$b^2 - b^2 = 0$$

$$V = \frac{a\pi}{b}\ [0] + \left[\frac{2\ (b^2 - 0^2)^{\frac{3}{2}}}{3}\right]$$

The values for St. Pauls Cathedral can be substituted into the general equation,

$$V = \frac{36\pi}{17}\int_0^{17} x\ (17^2 - x^2)^{\frac{1}{2}}\ dx$$

$$= -\frac{18\pi}{17}\left[\frac{2(17^2 - x^2)^{\frac{3}{2}}}{3}\right]_0^{17}$$

$$= \frac{18\pi}{17}\left[\frac{2(17^2 - 17^2)^{\frac{3}{2}}}{3}\right] + \left[\frac{2(17^2 - 0^2)^{\frac{3}{2}}}{3}\right]$$

$$= \frac{18\pi}{17}\ [0] + \left[\frac{2(17^2 - 0^2)^{\frac{3}{2}}}{3}\right]$$

$$V = 3468\pi\ \approx 11000\ m^3$$

The values and limits for St. Pauls Cathedral can be substituted into the equation,

$$V = \frac{46\pi}{21}\int_0^{21} x\ (21^2 - x^2)^{\frac{1}{2}}\ dx$$

$$= -\frac{23\pi}{21}\left[\frac{2(21^2 - x^2)^{\frac{3}{2}}}{3}\right]_0^{21}$$

$$= \frac{23\pi}{21}\left[\frac{2(21^2 - 21^2)^{\frac{3}{2}}}{3}\right] + \left[\frac{2(21^2 - 0^2)^{\frac{3}{2}}}{3}\right]$$

$$= \frac{23\pi}{21}\ [0] + \left[ \frac{2(21^2 - 0^2)^{\frac{3}{2}}}{3} \right]$$

$$V = 6762\pi \ \approx 21000\, m^3$$

These are both the same values found when using the disk method which makes the results more reliable.

To find the volume of Church of the Val-de-Grâce the equation of a hemisphere can be used,

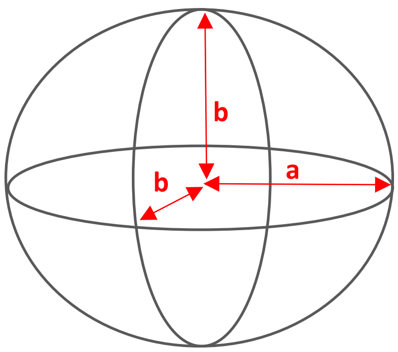
$$V = \frac{2}{3}\pi r^3$$

$$V = \frac{2}{3}\pi 11^3$$

$$V = \frac{2662}{3}\pi \ \approx 2800\, m^3$$

### Surface Area:

To work out the surface area of both St. Pauls Cathedral dome and St. Peter's Basilica dome, calculus is needed. The domes are rather rotated about the x-axis as the calculus does not require hyperbolic functions, therefore it is simpler. To rotate the dome, the major axis will be orientated horizontally instead of vertically, shown in the diagram below. The general equation must be altered to maintain the proportions, therefore it becomes, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , where b < a with its centre at (0,0), as the values a and b swap around.



Two formulas for surface area can be combined,

$$S = \int 2\pi y \, ds$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

Resulting in the complete formula,

$$S = 2\pi \int_0^a y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

y can be found:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2} (a^2 - x^2)$$

$$y = \frac{b}{a} \sqrt{a^2 - x^2} = \frac{b}{a} (a^2 - x^2)^{1/2}$$

$$\mathit{Only\ take\ the\ positive\ root\ to\ find\ the\ surface\ area\ of\ half\ an\ ellipsoid}$$

Use the chain rule to find the differentiation of y, $\frac{dy}{dx} = \frac{b}{a} \cdot \frac{1}{2}(-2x)(a^2 - x^2)^{-1/2}$

$$= -\frac{bx}{a\sqrt{a^2 - x^2}}$$

Or use implicit differentiation to find, $\frac{dy}{dx}$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\cdot\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)^2 = -\frac{b^4x^2}{a^4y^2}$$

These two equations are equal:

$$-\frac{b^2x}{a^2y} = -\frac{b^2x}{a^2\frac{b}{a}\sqrt{a^2-x^2}} \text{ , replace } y \text{ with its equation in terms of } a, b \text{ and } x$$

$$= -\frac{bx}{a\sqrt{a^2-x^2}} \text{ , cancel out the the squared values, giving the result when using the chain rule}$$

Plug the results into the equation,

$$S = 2\pi\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\left(\frac{-bx}{a\sqrt{a^2-x^2}}\right)^2}\,dx$$

$$= 2\pi\cdot\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}\,dx$$

$$\sqrt{v}\sqrt{u} = \sqrt{vu}$$

$$= 2\pi\cdot\frac{b}{a}\int_0^a \sqrt{(a^2-x^2)\left(1+\frac{b^2x^2}{a^2(a^2-x^2)}\right)}\,dx$$

*expand brackets*

$$= \frac{2\pi b}{a}\int_0^a \sqrt{\left(a^2-x^2+\frac{b^2x^2}{a^2}\right)}\,dx$$

*make $a^2$ the common denominator*

$$= \frac{2\pi b}{a}\int_0^a \sqrt{\frac{a^4-a^2x^2+b^2x^2}{a^2}}\,dx$$

$$= \frac{2\pi b}{a}\int_0^a \frac{1}{a}\sqrt{a^4-a^2x^2+b^2x^2}\,dx$$

$$= \frac{2\pi b}{a^2}\int_0^a \sqrt{a^4-(a^2-b^2)x^2dx}$$

$$= \frac{2\pi b}{a^2} \int_0^a a^2 \sqrt{1 - \frac{(a^2 - b^2)}{a^4} x^2 dx}$$

$$= 2\pi b \int_0^a \sqrt{1 - \frac{(a^2 - b^2)}{a^4} x^2 dx}$$

Use trig substitution for, $\frac{\sqrt{a^2-b^2}}{a^2} x$ , where $\frac{\sqrt{a^2-b^2}}{a^2} x = sin\theta$

$$\therefore x = \frac{a^2}{\sqrt{a^2 - b^2}} sin\theta$$

$$\therefore \frac{dx}{d\theta} = \frac{a^2}{\sqrt{a^2 - b^2}} cos\theta$$

$$\theta = arcsin\left(\frac{\sqrt{a^2 - b^2}}{a^2} x\right)$$

The limits therefore change where, x = a to $\theta = arcsin\left(\frac{a\sqrt{a^2-b^2}}{a^2}\right)$ and where x = 0, $\theta = 0$ as the integral is now with respect to d$\theta$.

$$= 2\pi b \int_0^{arcsin\left(\frac{a\sqrt{a^2-b^2}}{a^2}\right)} \sqrt{1 - sin^2\theta} \frac{a^2}{\sqrt{a^2 - b^2}} cos\theta \, d\theta$$

$$= \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}} \int_0^{arcsin\left(\frac{a\sqrt{a^2-b^2}}{a^2}\right)} \sqrt{1 - sin^2\theta} \, cos\theta \, d\theta$$

$$Pythagorean\ identity, cos^2\theta + sin^2\theta \equiv 1$$

$$1 - sin^2\theta \equiv cos^2\theta$$

$$= \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}} \int_0^{arcsin\left(\frac{a\sqrt{a^2-b^2}}{a^2}\right)} \sqrt{cos^2\theta} \, cos\theta \, d\theta$$

$$= \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}} \int_0^{arcsin\left(\frac{a\sqrt{a^2-b^2}}{a^2}\right)} cos^2\theta \ d\theta$$

$$= \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}} \int_0^{arcsin\left(\frac{a\sqrt{a^2-b^2}}{a^2}\right)} \frac{1 + cos2\theta}{2} \, d\theta$$

$$= \frac{a^2 b\pi}{\sqrt{a^2 - b^2}} \int_0^{arcsin\left(\frac{a\sqrt{a^2-b^2}}{a^2}\right)} (1 + cos2\theta) \ d\theta$$

$$= \frac{a^2 b\pi}{\sqrt{a^2 - b^2}} \left[\theta + \frac{1}{2} sin2\theta\right]_0^{arcsin\left(\frac{a\sqrt{a^2-b^2}}{a^2}\right)}$$

$$Double\ angle\ identity, sin2\theta \equiv 2sin\theta cos\theta$$

$$= \frac{a^2 b\pi}{\sqrt{a^2 - b^2}} \left[\theta + \frac{1}{2}(2sin\theta cos\theta)\right]_0^{arcsin\left(\frac{a\sqrt{a^2-b^2}}{a^2}\right)}$$

$$\sin(arcsin\ \theta) = \theta$$

$$\cos(arcsin\theta) = \sin(arccos\theta) = \sqrt{a^2 - \theta^2}$$

$$= \frac{a^2 b\pi}{\sqrt{a^2 - b^2}} \left[arcsin\left(\frac{a\sqrt{a^2 - b^2}}{a^2}\right) + \frac{a\sqrt{a^2 - b^2}}{a^2} \cdot \frac{\sqrt{a^4 - a^2(a^2 - b^2)}}{a^2} - (0)\right]$$

$$= \frac{a^2 b\pi}{\sqrt{a^2 - b^2}} \left[arcsin\left(\frac{\sqrt{a^2 - b^2}}{a}\right) + \frac{\sqrt{a^2 - b^2}\sqrt{a^4 - a^2(a^2 - b^2)}}{a^3}\right]$$

$$= \frac{a^2 b\pi}{\sqrt{a^2 - b^2}} \left[arcsin\left(\frac{\sqrt{a^2 - b^2}}{a}\right) + \frac{\sqrt{a^2 - b^2}\sqrt{a^4 - a^4 + a^2 b^2}}{a^3}\right]$$

$$= \pi \left[\frac{a^2 b}{\sqrt{a^2 - b^2}} arcsin\left(\frac{\sqrt{a^2 - b^2}}{a}\right) + \frac{a^2 b}{\sqrt{a^2 - b^2}} \cdot \frac{\sqrt{a^2 - b^2} \cdot ab}{a^3}\right]$$

$$= \pi \left[\frac{a^2 b}{\sqrt{a^2 - b^2}} arcsin\left(\frac{\sqrt{a^2 - b^2}}{a}\right) + \frac{b \cdot ab}{a}\right]$$

$$S = \pi\left[\frac{a^2 b}{\sqrt{a^2 - b^2}} \arcsin\left(\frac{\sqrt{a^2 - b^2}}{a}\right) + b^2\right]$$

Therefore, the surface area of St. Paul's Cathedral dome is,

$$S = \pi\left[\frac{18^2 17}{\sqrt{18^2 - 17^2}} \arcsin\left(\frac{\sqrt{18^2 - 17^2}}{18}\right) + 17^2\right]$$

$$S \approx 1900\ m^2$$

And the surface area of St. Peter's Basilica dome is,

$$S = \pi\left[\frac{23^2 21}{\sqrt{23^2 - 21^2}}\arcsin\left(\frac{\sqrt{23^2 - 21^2}}{23}\right) + 21^2\right]$$

$$S \approx 2900\, m^2$$

These two surface area results seem reasonable as the St. Peter's Basilica dome has larger a and b values which justify it having a larger surface area than St. Paul's Cathedral dome.

The surface area for the Church Val-de-Grâce dome can use simpler hemisphere formula, $A = 2\pi r^2$

$\therefore A = 2\pi 11^2 \ \therefore \approx 760m^2$

**Surface Area to Volume Ratio:**

By using a ratio, the domes can be fairly compared as the size of the domes do not affect the answer.

The S/V ratio for St. Pauls Cathedral dome is, $\frac{1887}{10895} \approx 0.1732$

The S/V ratio for St. Peter's Basilica dome is, $\frac{2948}{21243} \approx 0.1388$

The S/V ratio for Church Val-de-Grâce dome is, $\frac{2\pi r^2}{\frac{2}{3}\pi r^3} = \frac{6}{2r} = \frac{760}{2788} \approx 0.2726$

This determines that St. Peter's Basilica has the smallest surface area to volume ratio therefore has the highest heat efficiency.

# CONCLUSION

To conclude, not only is St. Peter's Basilica dome the most heat efficient, but both semi ellipsoids are more efficient than the hemisphere shaped dome. Furthermore, both semi ellipsoids have similar ratios with only a difference of 0.0344. This highlights their similarities not only geometrically, by both being semi ellipsoids but by having a similar S/V ratio. This can therefore suggest that Christopher Wren was more inspired by St. Peter's Basilica dome form. A limitation of the investigation is that I assumed the thickness of the material was constant, which can lead to values being either greater of smaller than the true value. Besides this, it can be concluded that semi ellipsoids are a more heat efficient shape than hemispheres. This information could be useful for picking a dome shape when designing a new dome. The investigation can therefore be extended to look at other three-dimensional shapes to find the most heat efficient building possible. The current conversation of global warming makes this topic very relevant as it can be a solution to minimise energy use. It could possibly integrate my own architectural designs for a greener and more sustainable future.

# Bibliography

Chan, Michael G. n.d. *Val-de-Grâce.* Accessed 03 14, 2021. https://www.thousandwonders.net/Val-de-Gr%C3%A2ce.

Ebert, Richard F. n.d. *Dome of St. Peter's Basilica in Rome, Italy.* Accessed 03 14, 2021. https://www.encirclephotos.com/image/dome-of-st-peters-basilica-in-rome-italy/.

Stack Exchange. 2016. *Deriving formula for surface area of an ellipsoid.* 12 9. Accessed 03 14, 2021. https://math.stackexchange.com/questions/2050158/deriving-formula-for-surface-area-of-an-ellipsoid.

Wikimedia. n.d. *File:Cathédrale St-Paul - coupole.jpg.* Accessed 03 14, 2021. https://commons.wikimedia.org/wiki/File:Cath%C3%A9drale_St-Paul_-_coupole.jpg.