MATHS EXPLORATION – DROP DEAD

INTRODUCTION

As I really enjoyed doing Probability in the Maths lessons, I decided to do my exploration on a probability problem. In class we only did relatively short problems, so now I want to try a longer one, as this might help me to investigate new approaches. I chose the dice problem *Drop Dead*, as it sounded interesting and challenging. I found this game in a pdf file about dice problems.[1]

THE GAME

Drop Dead is a game played with 5 standard dice. The aim is to get the highest score, however there is no skill involved. The player starts by throwing all of the dice. If they roll at least one 5 or 2, the player gets no points for this roll. This is because the 5s and 2s are 'dead' dice. Whenever there is at least one 'dead' dice in the current roll, the player obtains no score. The 'dead' dice are then removed, and the player rolls the left-over dice again. This process continues until all of the dice are 'dead'. The player only gets points for a roll if there are no 'dead' dice – he gets the sum of the numbers on the dice. The points are added together to give their score.

Example:

3, **2**, 1, **5**, 3 – Points: 0

The player gets 0 points, as they rolled two 'dead' dice (a 2 and a 5). These two dice are removed and the player rolls the three dice left.

6, 1, 4 – Points: 11

The player rolled no 'dead' dice, and so they gets a total of 11 points, which is the sum of the numbers rolled (6 + 1 + 4).

3, **2**, 4 – Points: 0

The player gets no points, as they rolled a 2.

1, 6 – Points: 7

There are no 'dead' dice, so the player gets 7 points for this roll.

3, **5** – Points: 0

The player rolled a 5, which is removed and means that the player obtains no score.

4 – Points: 4

The player gets 4 points.

6 – Points : 6

The player gets 6 points.

2 – Points : 0

The last die becomes 'dead'.

<u>Total</u> : 28

The points of each roll are added to give the score of the player for this round.

THE AIM OF THIS EXPLORATION IS TO FIND THE EXPECTED VALUE WHEN PLAYING THE GAME *DROP DEAD*.

[1] Conroy, M. M. (2014). *A Collection of Dice Problems.*

COLLECTING EXPERIMENTAL DATA

I played the game to get a feel for it. Here is a summary of my scores. In total I played 45 rounds.

0	0	31	28	8	0	0	0	21
4	0	40	47	45	14	16	0	4
4	8	30	18	26	0	22	39	0
18	0	6	46	0	10	17	10	6
6	0	22	36	1	24	3	9	17

The mean score is 636/45 = **14.13**

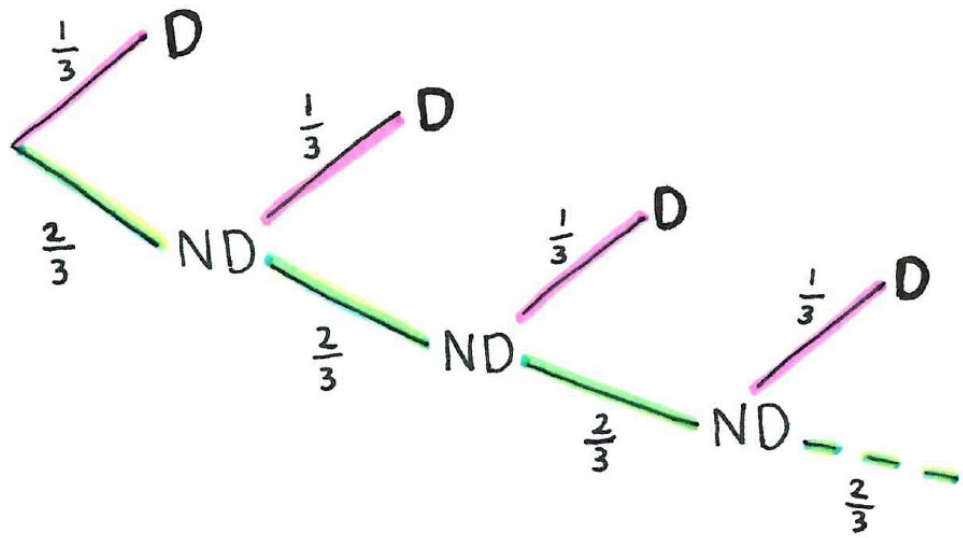
After having worked out the expected value, I can see how close the experimental data is.

FINDING THE EXPECTED SCORE WHEN PLAYING WITH 1 DIE

The problem by itself seemed really complicated to me, especially when I tried to draw a probability tree to represent the different outcomes. There are so many different possibilities that it becomes hard to work with. This is why I decided to simplify the problem by exploring what would happen if the game would be played with only one die instead of with five dice.

In one roll, the probability of the dice being 'dead' is $\frac{1}{3}$ and the probability of the dice not being 'dead' is $\frac{2}{3}$.

At first I drew a probability tree to represent the different outcomes and the probabilities of them happening. D represents a 'dead' die. ND represents that there is no 'dead' die.



Pink line: The die died (2 or 5), the score is 0 and the round is over

Green line: The die didn't die (1, 3, 4 or 6), the player receives a score of 3.5 and can roll again.

Dashed line: The pattern continues forever, as it will always be the same problem – what are the possible outcomes if there is one die left?

The expected score when rolling one die once, if it isn't dead, is (1+3+4+6)/4 = **3.5**

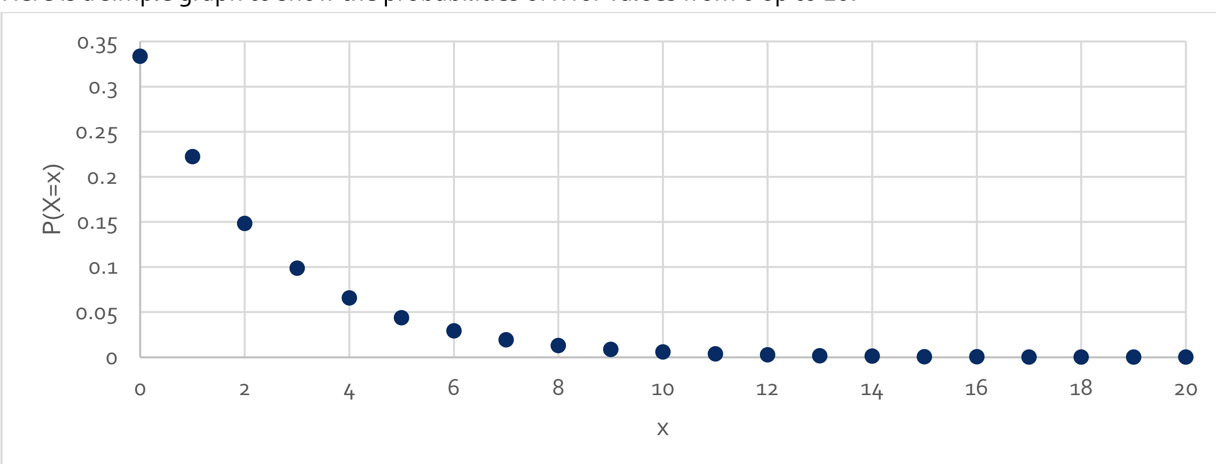
I made a table to show the probabilities of getting certain scores when playing the game once.

x	0	1	2	3	4	…	n
Score (S)	0	3.5	7	10.5	14		3.5n
P(X=x)	$\left(\frac{1}{3}\right)$	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2$	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3$	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4$		$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n$

x is the number of times the player throws the die until it 'dies' – so the number of throws for with a score is obtained.

The probability of obtaining a score of 3.5 in one roll is $\frac{2}{3}$. To get the probability of this score being the total of the round, this is multiplied by $\frac{1}{3}$, as the round needs to end with a 'dead' die. This is why to obtain the probability of getting a score for any multiple of 3.5, the probability $\frac{2}{3}$ is taken to the power of that multiple (the probability of getting a score of 3.5 AND getting a score of 3.5 again AND again…) and then multiplied by $\frac{1}{3}$ to end the round. This can be expressed as follows: P(S=3.5n) = $\left(\frac{2}{3}\right)^n \left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n$

Here is a simple graph to show the probabilities of x for values from 0 up to 20:



As x tends towards infinity, P(X=x) tends towards 0. This makes sense, as it is less likely to get a greater score.

Let X be the random variable representing the number of rolls before the die 'dies', and S be the random variable representing the score. S=3.5X

The expected score, E(S), is equal to the sum of |the scores times the probability of getting each score|. $E(S_1)$ is the expected score when playing with 1 die.

$E(S_1) = \sum_{n=0}^{\infty} SP$

S in an arithmetic sequence with a difference of 3.5. Therefore S(X=n) = 3.5n

P is a geometric sequence with a ratio of $\left(\frac{2}{3}\right)$. Therefore P(X=n) = $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n$

$E(1) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n \times 3.5n$

In order to solve this equation, I tried to find patterns using an Excel sheet. My aim is to find the sum of E(X) as x tends towards infinity. This can be written as: $\lim_{k\to\infty}\sum_{n=0}^{k}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n} \text{ x } 3.5\text{n}$

n	Score	Probability	E(X)	Ratio	Difference	
0	0	0.333333333				
1	3.5	0.222222222	0.777778	1.33333	-0.3333333	3
2	7	0.148148148	1.037037	1	-0.1111111	2
3	10.5	0.098765432	1.037037	0.88889	-0.0555556	1.666667
4	14	0.065843621	0.921811	0.83333	-0.0333333	1.5
5	17.5	0.043895748	0.768176	0.8	-0.0222222	1.4
6	21	0.029263832	0.61454	0.77778	-0.015873	1.333333
7	24.5	0.019509221	0.477976	0.7619	-0.0119048	1.285714
8	28	0.013006147	0.364172	0.75	-0.0092593	1.25
9	31.5	0.008670765	0.273129	0.74074	-0.0074074	1.222222
10	35	0.00578051	0.202318	0.73333	-0.0060606	1.2
11	38.5	0.003853673	0.148366	0.72727	-0.0050505	1.181818
12	42	0.002569116	0.107903	0.72222	-0.0042735	1.166667
13	45.5	0.001712744	0.07793	0.71795	-0.003663	1.153846
14	49	0.001141829	0.05595	0.71429	-0.0031746	1.142857
15	52.5	0.000761219	0.039964	0.71111	-0.0027778	1.133333
16	56	0.00050748	0.028419	0.70833	-0.002451	1.125
17	59.5	0.00033832	0.02013	0.70588	-0.0021786	1.117647
18	63	0.000225546	0.014209	0.7037	-0.0019493	1.111111
19	66.5	0.000150364	0.009999	0.70175	-0.0017544	1.105263
20	70	0.000100243	0.007017	0.7	-0.0015873	1.1
21	73.5	6.68286E-05	0.004912	0.69841	-0.001443	1.095238
22	77	4.45524E-05	0.003431	0.69697	-0.0013175	1.090909

However there was no pattern, as the series is clearly not an arithmetic nor a geometric series, and so I tried a different method. I typed the equation into my calculator, using 900 instead of ∞. The answer was exactly 7. As n tends towards infinity, the values added will get very close to zero, therefore I think that the final answer would be the same.

$E(S_1) = 7$

However this can also be shown more formally, using a geometric probability distribution.

$P(X=x) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{x}$

X = the number of rolls before a die 'dies'

This could be changed to $P(Y=y) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{y-1}$; y = 1, 2, 3…

Y = the total number of rolls in a round (including the one where the die 'dies')

$Y \sim Geo\left(\frac{2}{3}\right)$

$$E(X) = \left(\frac{1}{1-p}\right) = \left(\frac{1}{1-\frac{2}{3}}\right) = \left(\frac{1}{\frac{1}{3}}\right) = 3$$

This means that there will be 3 rolls in total. As no score is obtained for the third one, when the die 'dies', the expected number of rolls for which a score is obtained is two. So $E(X_1)=2$. As S = 3.5X, $E(S_1) = 2 \times 3.5 = 7$.

In other words: the expected score when playing the game with one die is 7.

FINDING THE EXPECTED SCORE WHEN PLAYING WITH 2 DICE

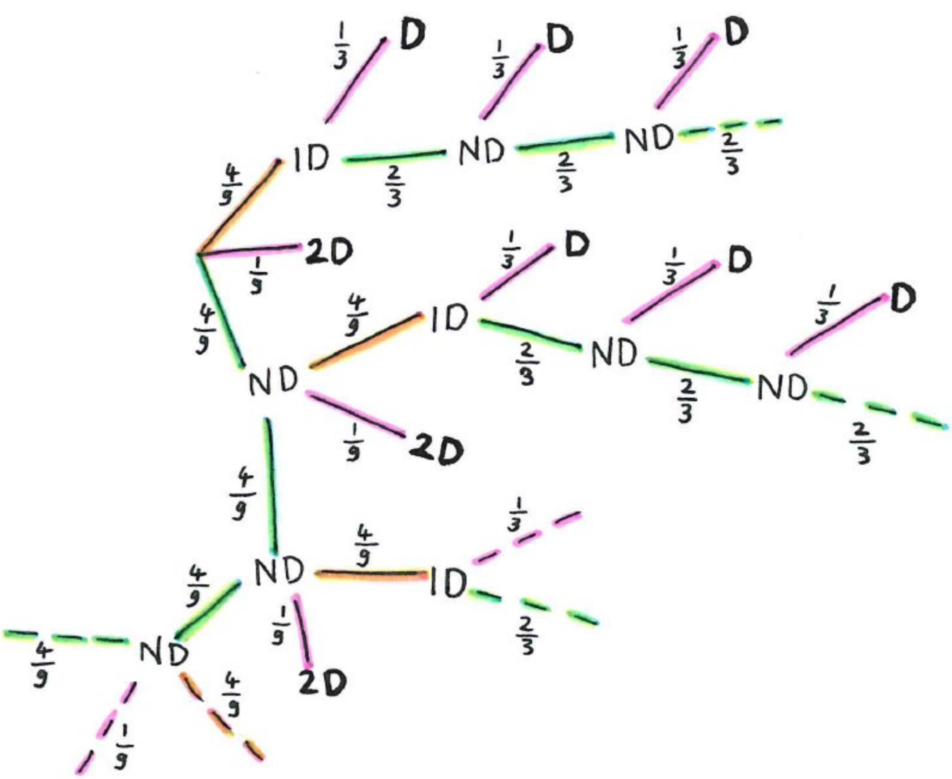
Now that I have solved the problem for one die, I will try to solve it for two. I hope that there will be a pattern that will bring me closer to the final solution.

When playing with 2 dice, there are 3 different outcomes: 2 Dead, 1 Dead, No Dead.

The probabilities of getting each outcome are shown in the following table.

2 Dead	$\left(\frac{1}{3}\right) \times \left(\frac{1}{3}\right)$	$\frac{1}{9}$
1 Dead	$\left(\frac{1}{3}\right) \times \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right) \times \left(\frac{1}{3}\right)$	$\frac{4}{9}$
No Dead	$\left(\frac{2}{3}\right) \times \left(\frac{2}{3}\right)$	$\frac{4}{9}$

Again, I drew a probability tree to represent the different outcomes.



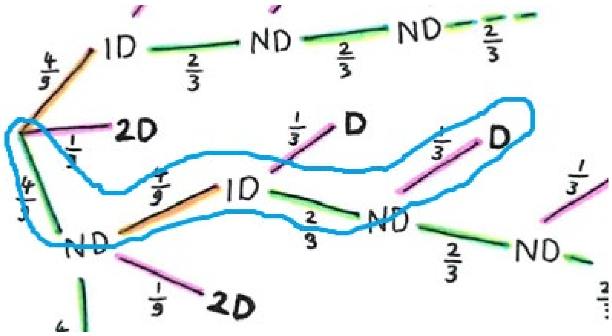
Pink line: The die died (2 or 5), the score is 0 and the round is over

Green line: The dice didn't die (1, 3, 4 or 6), the player receives a score of 7 or 3.5 and can roll again.

Orange line: One of the two dice died, so the player doesn't receive a score, but the game continues with 1 die

Dashed line: The pattern continues forever

As this probability tree looks more complicated than the first one, I will explain an example of a possible path:



In this blue path starts with a roll in which there are no 'dead' dice. This means that the player gets a score and can roll both of the dice again. In the next roll, they roll one 'dead' die, so they won't get a score, however they continue to play with the other die. Then, they roll a non-'dead' die, get a score and can roll the die again. Next, they roll 'dead' die, which means that the round is finished and the scores are added up.

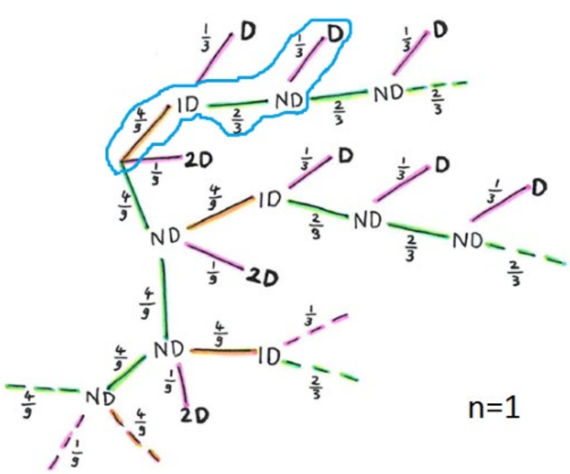
I made a table to show the probabilities of getting certain scores.

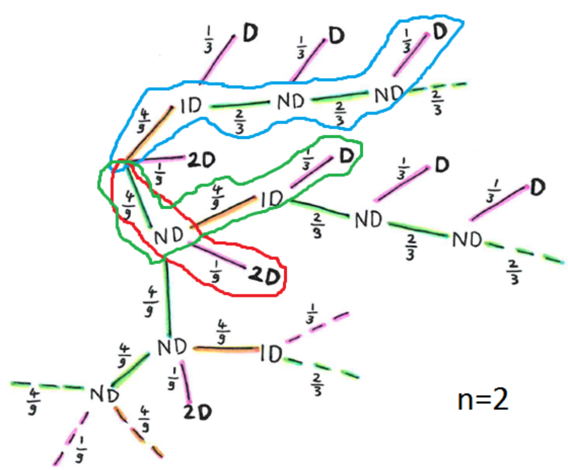
x	0	1	2	3
Score (S)	0	3.5	7	10.5
P(X=x)	$\left(\frac{1}{9}\right)$	$\left(\frac{4}{9}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)$	$\left(\frac{4}{9}\right)\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)+\left(\frac{4}{9}\right)\left(\frac{1}{9}\right)+\left(\frac{4}{9}\right)^2\left(\frac{1}{3}\right)$	$\left(\frac{4}{9}\right)\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)+\left(\frac{4}{9}\right)^2\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)$

4	5
14	17.5
$\left(\frac{4}{9}\right)\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)+\left(\frac{4}{9}\right)^2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)+\left(\frac{4}{9}\right)^2\left(\frac{1}{9}\right)+\left(\frac{4}{9}\right)^3\left(\frac{1}{3}\right)$	$\left(\frac{4}{9}\right)\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)+\left(\frac{4}{9}\right)^2\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)+\left(\frac{4}{9}\right)^3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)$

n
3.5n
$\left(\frac{4}{9}\right)\left(\frac{2}{3}\right)^n\left(\frac{1}{3}\right)+\left(\frac{4}{9}\right)^2\left(\frac{2}{3}\right)^{n-2}\left(\frac{1}{3}\right)+\left(\frac{4}{9}\right)^{\frac{n}{2}}\left(\frac{1}{9}\right)+\left(\frac{4}{9}\right)^{\frac{n}{2}+1}\left(\frac{1}{3}\right)\ldots$

In order to make this clearer, I drew the possible possibilities for n=1 and n=2 on a probability tree





When there is more than one way of obtaining a score, the probabilities of the different paths are added together.

From this table, in order to find the expected score, I would have to what I did when solving the problem for one die: find the sum of (3.5n x the formula for n), when substituting numbers in for x from 0 to infinity.

This is why I tried to find a pattern for n, however as it differs for odd and even values of n, and new series are added each time x increases by 1, this was very complicated. As for 3 dice this would get even more complicated and probably impossible to solve, I have to find another way of finding the expected value.

A DIFFERENT APPROACH TO THE PROBLEM

I noticed that I started both of the problems above by calculating the probabilities of the different outcomes. So the probability of getting a certain number of dead dice when rolling a certain number of dice. The formulas to find the expected score consisted only of these probabilities.

This is why I decided to make a table for the probabilities of getting a certain number of dead dice out of a certain number of dice rolled. The number of dice rolled is *n*. The number of dead dice is *d*. The probability of getting *d* dead dice when rolling *n* dice is $P_{n/d}$

$$P_{n/d} = nCd\left(\frac{1}{3}\right)^{d}\left(\frac{2}{3}\right)^{n-d}$$

$\frac{1}{3}$ is the probability of one dice being dead (*d*) and $\frac{2}{3}$ is the probability of one dice not being dead (*n*-*d*). Their probabilities are taken to the power of the number of dead and non-dead dice. These two parts are then multiplied by nCd as there are several combinations. For example when 3 dice are rolled, there are three different combinations of obtaining 2 dead dice: [Dead, non-dead, dead]; [dead, dead, non-dead]; [non-dead, dead; dead].

Number of dice	Dead dice	0	1	2	3	4	5
1		$\frac{2}{3}$	$\frac{1}{3}$	0	0	0	0
2		$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	0	0	0
3		$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$	0	0
4		$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$	0
5		$\frac{32}{243}$	$\frac{80}{243}$	$\frac{80}{243}$	$\frac{40}{243}$	$\frac{10}{243}$	$\frac{1}{243}$

This should help me find the expected score when playing the game with a certain number of die. However I couldn't work out how. In the pdf file containing the problem and its solution I found this equation[2]:

[2] Conroy, M. M. (2014). *A Collection of Dice Problems.*

$$E(1) = 3.5P_{1,1} + E(1)P_{1,1}.$$

As they used *j* in order to represent the number on non-dead dice, $P_{1,1}$ would be the same as $P_{1/0}$ in my table.

This can be rearranged:

$E(S_1) - E(S_1)P_{1/0} = 3.5P_{1/0}$

$E(S_1)\ (1-P_{1/0}) = 3.5P_{1/0}$

$E(S_1) = \frac{3.5P1/0}{1-P1/0}$

$E(S_1) = \frac{(3.5)\frac{2}{3}}{1-\frac{2}{3}} = \frac{3.5 \text{ x } 6}{3}$

$E(S_1) = 7$

This is the same value that I got for $E(S_1)$.

Therefore the equation seems to work. But I couldn't find the pattern. It only became clear to me after I also saw the equation for the expected value when playing with two dice.

$$E(2) = (2 \cdot 3.5 + E(2))\, P_{2,2} + E(1)P_{2,1}.$$

Here is how it would be written using my notation:

$E(S_2) = (2 \times 3.5 + E(S_2))P_{2/0} + E(S_1)P_{2/1}$

$E(S_2) = (7 + E(S_2))\ \frac{4}{9} + 7 \times \frac{4}{9}$

$E(S_2) = \frac{28}{9} + \frac{4}{9}E(S_2) + \frac{28}{9}$

$E(S_2) - \frac{4}{9}E(S_2) = \frac{56}{9}$

$E(S_2) = \frac{56}{9\left(1-\frac{4}{9}\right)} = \frac{56}{9-4} = \frac{56}{5}$

$E(S_2) = 11.2$

From these two equations I can deduce the equations for the other expected scores up to 5. In order to work out $E(S_5)$ I need to know the values of $E(S_3)$ and $E(S_4)$, meaning that I have to work them out.

Finding the expected score when rolling 3 dice:

$E(S_3) = (3 \times 3.5 + E(S_3))P_{3/0} + E(S_2)P_{3/1} + E(S_1)P_{3/2}$

$E(S_3) = (10.5 + E(S_3))\,\frac{8}{27} + 11.2\left(\frac{12}{27}\right) + 7\left(\frac{6}{27}\right)$

$E(S_3) - \frac{8}{27}E(S_3) = \frac{28}{9} + \frac{224}{45} + \frac{14}{9} = \frac{434}{45}$

$E(S_3) = \frac{434}{45\left(1-\frac{8}{27}\right)}$

$E(S_3) = \frac{1302}{95} \approx 13.7$

Finding the expected score when rolling 4 dice:

$E(S_4) = (4 \times 3.5 + E(S_4))P_{4/0} + E(S_3)P_{4/1} + E(S_2)P_{4/2} + E(S_1)P_{4/3}$

$E(S_4) = (14 + E(S_4))\,\frac{16}{81} + \left(\frac{1302}{95}\right)\left(\frac{32}{81}\right) + 11.2\left(\frac{24}{81}\right) + 7\left(\frac{8}{81}\right)$

$E(S_4) - \frac{16}{81}E(S_4) = \frac{224}{81} + \left(\frac{1302}{95}\right)\left(\frac{32}{81}\right) + \frac{448}{135} + \frac{56}{81}$

$$E(S_4) = \frac{\frac{224}{81} + \left(\frac{1302}{95}\right)\left(\frac{32}{81}\right) + \frac{448}{135} + \frac{56}{81}}{1-\frac{16}{81}}$$

$E(S_4) \approx 15.2$

Finding the expected score when rolling 5 dice:

$E(S_5) = (5 \times 3.5 + E(S_5))P_{5/0} + E(S_4)P_{5/1} + E(S_3)P_{5/2} + E(S_2)P_{5/3} + E(S_1)P_{5/4}$

$$E(S_5) = (17.5 + E(S_5))\,\frac{32}{243} + \left(\frac{\frac{224}{81} + \left(\frac{1302}{95}\right)\left(\frac{32}{81}\right) + \frac{448}{135} + \frac{56}{81}}{1-\frac{16}{81}}\right)\left(\frac{80}{243}\right) + \left(\frac{1302}{95}\right)\left(\frac{80}{243}\right) + 11.2\left(\frac{40}{243}\right) + 7\left(\frac{10}{243}\right)$$

$E(S_5) - \frac{32}{243}E(S_5) = \frac{560}{243} + \frac{300160}{60021} + \frac{6944}{1539} + \frac{448}{243} + \frac{70}{243}$

$$E(S_5) = \frac{\frac{560}{243} + \frac{300160}{60021} + \frac{6944}{1539} + \frac{448}{243} + \frac{70}{243}}{1-\frac{32}{243}}$$

$E(S_5) \approx 16.1$

CONCLUSION

I met the aims of my exploration by finding the expected score when playing the game *Drop Dead*. When rounded to one decimal place, $E(S_5)$ is equal to 16.1. This is slightly higher than 14.13, the mean score I got when playing right at the beginning. Therefore it could be said that I was unlucky.

During the exploration, I tried several different approaches until I finally got to the answer. This process helped me to get a more thorough understanding of the problem. I learned how to find the solution to a longer and harder problem by breaking it up into small parts – by starting with one dice, then two, etc.

If I would want to extend the problem, I could change several factors. For example, if the dice would be dead when a 1 or 2 are rolled instead of a 2 and a 5. This would be easy to adapt, as the probabilities wouldn't change. For more of a challenge, I could try to find the expected score when playing with 20 dice. This would mean that I would need a different quicker method, which would allow me to work out $E(S_{20})$ without working out all of the expected scores before. Or I could try working out the expected score for the same game, but with 5 biased dice.

BIBLIOGRAPHY

Conroy, M. M. (2014). *A Collection of Dice Problems.*