Investigating where to cut circular and elliptic shaped cheeses so that the pieces are the same size

Introduction :

Cheese is a popular food all around the world with hundreds of different kinds. When sold they usually have a circular shape or sometimes an elliptic shape.

When cutting a circular cheese such as Camembert, we usually slice it into sectors which makes it relatively simple to cut pieces that are the same size. However, when cutting an elliptic shaped cheese like Caprice des Dieux we cut parallel slices along it. This makes it considerably harder to tell, due to its shape, whether there is an equal volume of cheese in each slice.

Figure 1 An image of a Camembert cheese

Figure 2 An image of a Caprice des Dieux cheese

Yet as long as the cheese in question is a prism with constant cross section, by looking at the surface area of the top surface I could use integration for finding the area under a graph to find where to cut so that the surface area of each piece of cheese is the same.

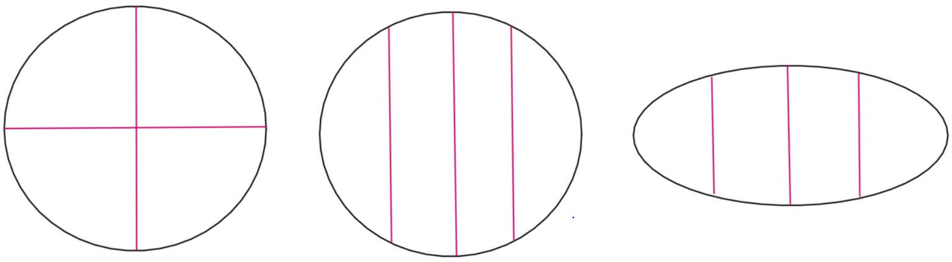

Figure 1 Diagram to illustrate the different ways of cutting a circle and ellipse into four slices

The aim of my exploration is to work out where to cut a Caprice des Dieux and a Camembert with parallel slices such that the resulting pieces of cheese are the same size.

When sharing cheese with my parents we frequently find ourselves debating over whether the pieces are the same size. This got me wondering how I could work out exactly where to cut my parallel slices on either a circular cheese or an elliptic shaped one to try and get pieces of equal size.

As such, I thought that in this exploration I could use integration to try and work out where to cut my favourite cheeses if I want there to be three equal sized pieces that are cut with parallel slices.

Furthermore, integration is one of my favourite IB level maths topics that we have done so I thought this investigation would be perfect for using the knowledge that I have gained so far in this topic.

Through this exploration I hope to work out how to find the area under a part of a circle and an ellipse using my knowledge of integration.

Exploration:

Looking at the circle

To see if my maths IA idea would work I first decided to try with a circle as I thought that this might be easier than the ellipse. It will be divided into three pieces.

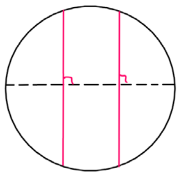

Figure 2 Diagram to illustrate how the circle should be cut

The general equation for a circle where the centre is at the origin is: $\mathbf{x^2 + y^2 = r^2}$

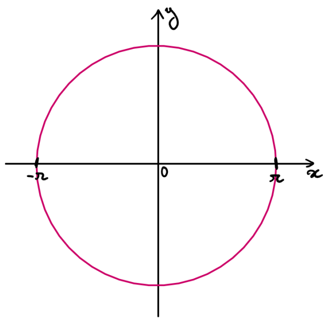

Figure 3 Diagram showing the graph of the function $x^2+y^2=r^2$

The area of the circle is πr^2.

The general equation needs to be rearranged with y on one side so that I can integrate with respect to x.

$$y^2 + x^2 = r^2$$

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

As both halves of a circle are identical I only need a function for half of the circle so that I can work out the area under the graph.

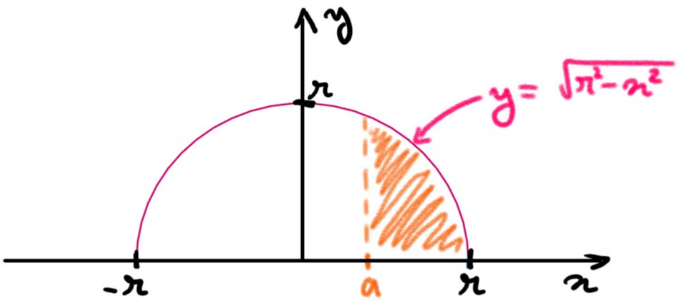

Figure 4 Diagram showing the graph of the function for half a circle

As I am only cutting the circle three times, there are only two slices at x = a and at x = -a.

As I am only working out the area under half a circle then I must halve the original area so that it becomes $\frac{\pi r^2}{2}$. The area of each slice will then be $\frac{\pi r^2}{6}$.

Integrating the equation for a half circle

I want to find a so that the area under the graph from x = r to x = a is equal to $\frac{\pi r^2}{6}$.

$\int_a^r \sqrt{r^2 - x^2}\, dx = \frac{\pi r^2}{6}$

I can integrate $\sqrt{r^2 - x^2}$ using integration by parts:

$$\int u \cdot \frac{dv}{dx} dx \quad = \quad uv \quad - \quad \int v \cdot \frac{du}{dx} dx$$

Figure 5 The formula for integrating by parts

This allows me to differentiate $\sqrt{r^2 - x^2}$ instead which is much easier as I do not know how to integrate it directly.

$$\int 1 \times \sqrt{r^2 - x^2}\, dx$$

$u = \sqrt{r^2 - x^2}$ $\qquad \frac{du}{dx} = -x(r^2 - x^2)^{\frac{-1}{2}}$

$\frac{dv}{dx} = 1$ $\qquad v = x$

This means:

$\int 1 \times \sqrt{r^2 - x^2}\, dx = x\sqrt{r^2 - x^2} - \int x \times \frac{-x}{\sqrt{r^2-x^2}} dx$

$$= x\sqrt{r^2 - x^2} - \int -1 \times \frac{x^2}{\sqrt{r^2-x^2}}\, dx$$

$$= x\sqrt{r^2 - x^2} + \int \frac{x^2}{\sqrt{r^2-x^2}}\, dx$$

At first, I was unsure how to work out $\int \frac{x^2}{\sqrt{r^2-x^2}}\, dx$ but then I remembered a similar example that we did in a Higher Level maths lesson where the equation was in a similar form and we used integration by substitution where we used $\sin\theta$. So I realised I will need to use integration by substitution.

I will use the substitution $x = r\sin\theta$.

$$\frac{dx}{d\theta} = r\cos\theta$$

$$= x\sqrt{r^2 - x^2} + \int \frac{r^2\sin^2\theta}{\sqrt{(r^2 - r^2\sin^2\theta)}} \times r\cos\theta\, d\theta$$

$$= x\sqrt{r^2 - x^2} + \int \frac{r^2\sin^2\theta \times r\cos\theta}{\sqrt{(r^2(1-\sin^2\theta)}}\, d\theta$$

Thanks to double angle identities we know that $1 - \sin^2\theta \equiv \cos^2\theta$ so I can replace this in the denominator.

$$= x\sqrt{r^2 - x^2} + \int \frac{r^2\sin^2\theta \times r\cos\theta}{r\sqrt{\cos^2\theta}}\, d\theta$$

$$= x\sqrt{r^2 - x^2} + \int \frac{r^2\sin^2\theta \times r\cos\theta}{r\cos\theta}\, d\theta$$

The denominator cancels out with $r\cos\theta$ in the numerator.

$$= x\sqrt{r^2 - x^2} + \int r^2\sin^2\theta\, d\theta$$

$$= x\sqrt{r^2 - x^2} + r^2\int \sin^2\theta\, d\theta$$

We know that $\sin^2\theta \equiv \frac{1}{2} - \frac{1}{2}\cos\theta$ so I can replace $\sin^2\theta$ with $\frac{1}{2} - \frac{1}{2}\cos 2\theta$.

$$= x\sqrt{r^2 - x^2} + r^2 \int (\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta)\ \mathrm{d}\theta$$

I can now integrate this much more easily.

$$= x\sqrt{r^2 - x^2} + r^2(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta + c)$$

$$= x\sqrt{r^2 - x^2} + \tfrac{r^2}{2}\theta - \tfrac{r^2}{4}\sin 2\theta + C_1$$

We know the double angle identity $\sin 2\theta \equiv 2\sin\theta\cos\theta$.

$$= x\sqrt{r^2 - x^2} + \tfrac{r^2}{2}\theta - \tfrac{r^2}{4} 2\sin\theta\cos\theta + C_1$$

The next step is to change the equation so that it only uses x. ($C_1$is a constant)

I used the substitution $x = r\sin\theta$ so $\theta = \sin^{-1}\frac{x}{r}$.

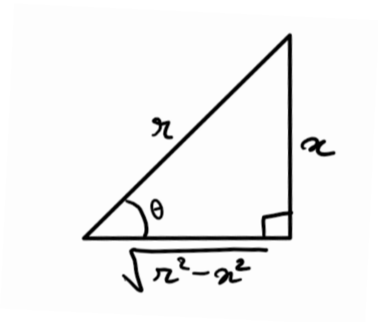

Figure 6 A diagram of a triangle to work out cosθ

By using Pythagoras's theorem and my knowledge of trigonometry I can find that $\cos\theta = \frac{\sqrt{r^2 - x^2}}{r}$.

$$= x\sqrt{r^2 - x^2} + \tfrac{r^2}{2}(\sin^{-1}\tfrac{x}{r}) - \tfrac{2r^2}{4} \times \tfrac{x}{r} \times \tfrac{\sqrt{r^2 - x^2}}{r} + C_1$$

$$= x\sqrt{r^2 - x^2} + \tfrac{r^2}{2}(\sin^{-1}\tfrac{x}{r}) - \tfrac{x\sqrt{r^2 - x^2}}{2} + C_1$$

$$= \tfrac{r^2}{2}(\sin^{-1}\tfrac{x}{r}) + \tfrac{x\sqrt{r^2 - x^2}}{2} + C_1$$

Now that I have integrated $\int \sqrt{r^2 - x^2}\, dx$ I can work out $\int_a^r \sqrt{r^2 - x^2}\, dx$.

$$\int_a^r \sqrt{r^2 - x^2}\, dx = \left[\tfrac{r^2}{2}(\sin^{-1}\tfrac{x}{r}) + \tfrac{x\sqrt{r^2 - x^2}}{2}\right]_a^r$$

$$= (\frac{r^2}{2}(\sin^{-1}\frac{r}{r}) + \frac{r\sqrt{r^2-r^2}}{2}) - (\frac{r^2}{2}(\sin^{-1}\frac{a}{r}) + \frac{a\sqrt{r^2-a^2}}{2})$$

$$= \left(\frac{r^2\pi}{4} + 0\right) - (\frac{r^2}{2}(\sin^{-1}\frac{a}{r}) + \frac{a\sqrt{r^2-a^2}}{2})$$

To find a, I have to solve this:

$$\left(\frac{r^2\pi}{4} + 0\right) - (\frac{r^2}{2}(\sin^{-1}\frac{a}{r}) + \frac{a\sqrt{r^2-a^2}}{2}) = \frac{\pi r^2}{6}$$

<u>Working out a for the dimensions of a Camembert</u>

I measured a Camembert and found that it has a diameter of 10.2cm so the radius r is equal to 5.1cm.

The area of one slice from x=5.1 to x=a is :

$$\left(\frac{5.1^2\pi}{4} + 0\right) - \left(\frac{5.1^2}{2}\left(\sin^{-1}\frac{a}{5.1}\right) + \frac{a\sqrt{5.1^2-a^2}}{2}\right) = \frac{\pi \times 5.1^2}{6}$$

To find a all I need to do is put both $\left(\frac{5.1^2\pi}{4}\right) - (\frac{5.1^2}{2}(\sin^{-1}\frac{a}{5.1}) + \frac{a\sqrt{5.1^2-a^2}}{2})$ and $\frac{\pi \times 5.1^2}{6}$ into GeoGebra separately. Where they intersect will give me a.

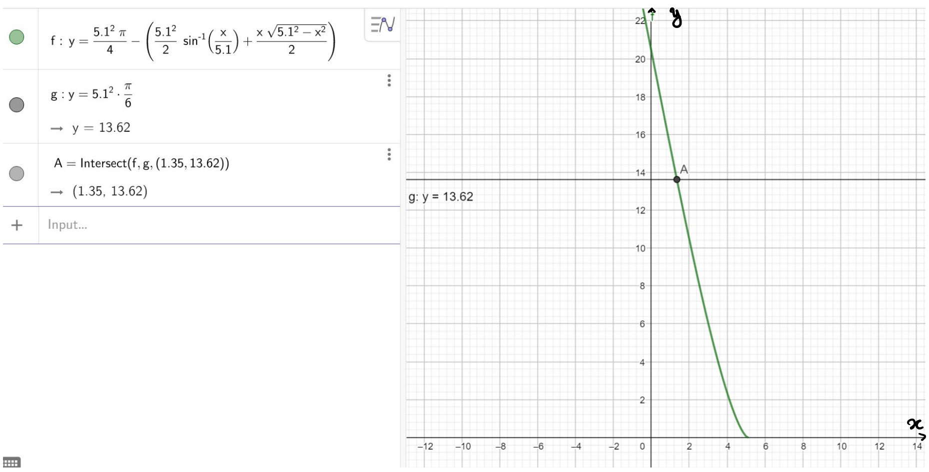

Figure 7 A screen shot of GeoGebra to show the intersection of the graphs

This gives me a≈1.35

I checked that this value was correct by inputting the definite integral $\int_{1.35}^{5.1}\sqrt{5.1^2 - x^2}\,dx$ into my calculator and this gave me the correct result.

My calculator returns approximately the value 13.6 which is close to $\frac{5.1^2\pi}{6}$ meaning that this is the correct result.

I then checked the result for the areas of the other two slices which I get from $\int_{-1.35}^{1.35} \sqrt{5.1^2 - x^2}\, dx$ and $\int_{-5.1}^{-1.35} \sqrt{5.1^2 - x^2}\, dx$. Both also return approximately 13.6.

$5.1 - 1.35 \approx 3.75$

This means that:

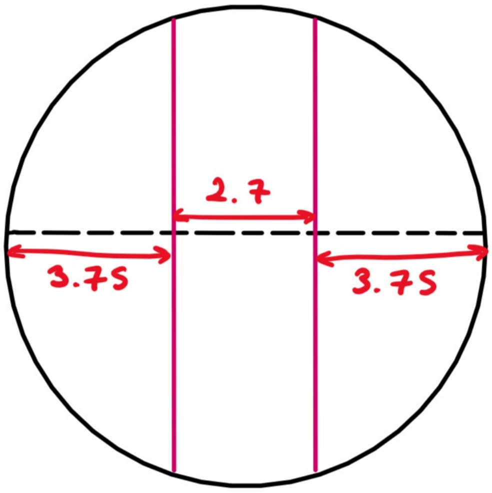

Figure 8 Diagram to show the results of the exploration

The cheese should be cut 3.75cm along the diameter and then 2.7cm along from this for the second slice.

Looking at the ellipse

Since it worked well for the circular cheese I decided to try this approach again but this time for an ellipse which will be sliced into three pieces also.

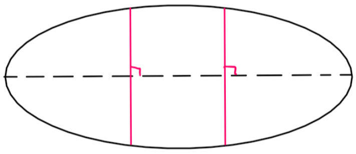

Figure 9 Diagram to illustrate how the ellipse should be cut

The general equation for an ellipse is: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The area of the ellipse is abπ.

The general equation needs to be rearranged with y on one side so that I can integrate with respect to x.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right)$

$y = b\sqrt{1 - \frac{x^2}{a^2}}$

Like with the circle, this will only give me half of the ellipse.

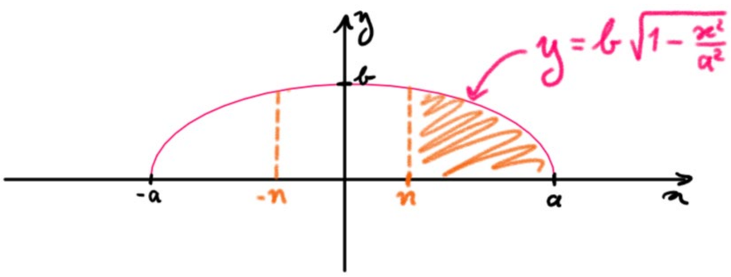

Figure 10 Diagram showing the graph of the function for half an ellipse

Here, the slices are at x=n and x=-n.

I halve the area of the ellipse to get $\frac{ab\pi}{2}$ and each slice will therefore have an area of $\frac{ab\pi}{6}$.

Integrating the equation for half of an ellipse

I want to find n so that the area under the graph from x = a to x = n is equal to $\frac{ab\pi}{6}$.

$$\int_n^a b\sqrt{}(1 - \frac{x^2}{a^2})\, dx = \frac{ab\pi}{6}$$

I can rearrange this a little to make it easier to integrate.

$$\int b\sqrt{1 - \frac{x^2}{a^2}}\, dx = b\int (1 - \frac{x^2}{a^2})^{\frac{1}{2}}\, dx$$

$$= b\int \frac{(a^2-x^2)^{\frac{1}{2}}}{a}\, dx$$

$$= \frac{b}{a}\int (a^2 - x^2)^{\frac{1}{2}}\, dx$$

I can now use integration by parts on this.

$$\int u \cdot \frac{dv}{dx} dx \quad = \quad uv \quad - \quad \int v \cdot \frac{du}{dx} dx$$

Figure 11 The formula for integrating by parts

$u = (a^2 - x^2)^{\frac{1}{2}}$ $\qquad$ $\frac{dv}{dx} = 1$

$\frac{du}{dx} = -x(a^2 - x^2)^{-\frac{1}{2}}$ $\qquad$ $v = x$

This means:

$$\frac{b}{a}\int (a^2 - x^2)^{\frac{1}{2}}\, dx = \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} - \int x\left(-x(a^2 - x^2)^{-\frac{1}{2}}\right) dx$$

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} - \int \frac{-x^2}{\sqrt{a^2-x^2}} dx$$

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + \int \frac{x^2}{\sqrt{a^2-x^2}} dx$$

To integrate $\int \frac{x^2}{\sqrt{a^2-x^2}} dx$ I can use integration by substitution with the substitution $x = a\sin\theta$.

$\frac{dx}{d\theta} = a\cos\theta$

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + \int \frac{a^2\sin^2\theta}{\sqrt{a^2 - a^2\sin^2\theta}} a\cos\theta\, dx)$$

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + \int \frac{a^2\sin^2\theta \times a\cos\theta}{(a^2(1-\sin^2\theta))^{\frac{1}{2}}} dx)$$

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + \int \frac{a^2\sin^2\theta \times a\cos\theta}{a(1-\sin^2\theta)^{\frac{1}{2}}} dx)$$

Using the Pythagorean identity we know that $1 - \sin^2\theta \equiv \cos^2\theta$ so I can replace this in the denominator.

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + \int \frac{a^2\sin^2\theta \times a\cos\theta}{a(\cos^2\theta)^{\frac{1}{2}}} dx)$$

The denominator cancels out with $a\cos\theta$ in the numerator.

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + \int a^2\sin^2\theta\, dx)$$

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + a^2\int \sin^2\theta\, dx)$$

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + a^2\int \frac{1}{2} - \frac{\cos 2\theta}{2} d\theta$$

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + a^2(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + c)$$

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}a^2(\theta - \frac{1}{2}\sin 2\theta + c)$$

The double angle identity $\sin 2\theta = 2\sin\theta\cos\theta$ means that I can replace $\frac{1}{2}\sin 2\theta$ with $\sin\theta\cos\theta$.

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}a^2(\theta - \sin\theta\cos\theta + C_1)$$

The next step is to change to equation so that it only uses x.

I used the substitution $x = a\sin\theta$ so $\frac{x}{a} = \sin\theta$ and $\theta = \sin^{-1}\frac{x}{a}$.

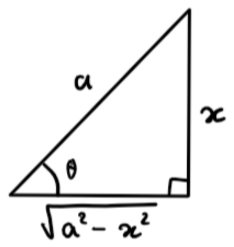

Figure 12 A diagram of a triangle to work out cosϑ

Using this triangle I know that $\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$.

$$= \frac{b}{a}(x(a^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}a^2(\sin^{-1}\frac{x}{a} - \frac{x}{a} \times \frac{\sqrt{a^2-x^2}}{a} + C_1)$$

$$= \frac{b}{a}(\frac{x}{2}(a^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a} + C_1)$$

Now that I have integrated $\int b\sqrt{1-\frac{x^2}{a^2}}\,dx$ I can work out $\int_n^a b\sqrt{(1-\frac{x^2}{a^2})}\,dx$.

$$\int_n^a b\sqrt{(1-\frac{x^2}{a^2})}\,dx = \frac{b}{a}[\frac{x}{2}(a^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a})]_n^a$$

$$= \frac{b}{a}((\frac{a}{2}\sqrt{a^2-a^2} + \frac{1}{2}a^2\sin^{-1}\frac{a}{a}) - (\frac{n}{2}\sqrt{a^2-n^2} + \frac{1}{2}a^2\sin^{-1}\frac{n}{a}))$$

$$= \frac{b}{a}((0 + \frac{1}{4}a^2\pi) - (\frac{n}{2}\sqrt{a^2-n^2} + \frac{1}{2}a^2\sin^{-1}\frac{n}{a}))$$

<u>Working out n for the dimensions of a Caprice de Dieux</u>

Now I want to find n so that the area under the graph is equal to $\frac{ab}{6}\pi$.

$$\frac{b}{a}(\frac{1}{4}a^2\pi - (\frac{n}{2}\sqrt{a^2-n^2} + \frac{1}{2}a^2\sin^{-1}\frac{n}{a})) = \frac{ab}{6}\pi$$

To do this I first need to replace a and b with the dimensions for a Caprice des Dieux.

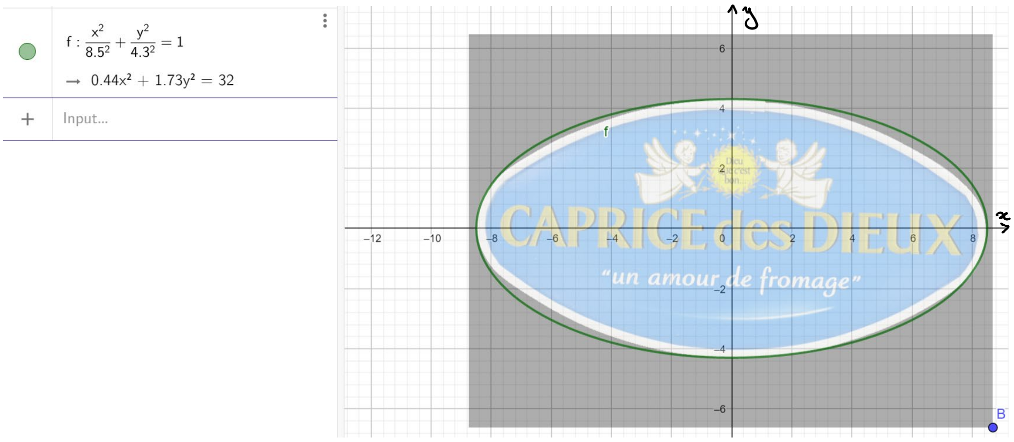

Figure 13 A GeoGebra graph showing the closest ellipse to the shape of a Caprice des Dieux

I measured a Caprice des Dieux and found that it has a length of 16.99cm and a width of 8.61cm. As shown in figure 15, the cheese does not have an exact elliptical shape but is close enough that I considered that it should work.

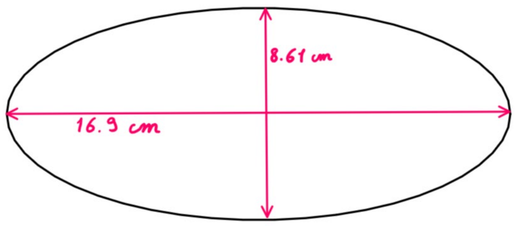

Figure 14 Diagram showing the dimensions of a Caprice des Dieux

This means that $a = 16.9 \div 2 \approx 8.5$ and $b = 8.61 \div 2 \approx 4.3$.

$$\frac{4.3}{8.5}(\frac{1}{4}8.5^2\pi - (\frac{n}{2}\sqrt{8.5^2 - n^2} + \frac{1}{2}8.5^2(\sin^{-1}\frac{n}{8.5})) = \frac{8.5 \times 4.3}{6}\pi$$

To find n I need to put both $\frac{4.3}{8.5}(\frac{1}{4}8.5^2\pi - (n\sqrt{8.5^2 - n^2} + \frac{1}{2}8.5^2(\sin^{-1}\frac{n}{8.5} - \frac{n\sqrt{8.5^2-n^2}}{8.5^2}))$ and $\frac{8.5 \times 4.3}{6}\pi$ into GeoGebra separately so that I can find where they intersect.

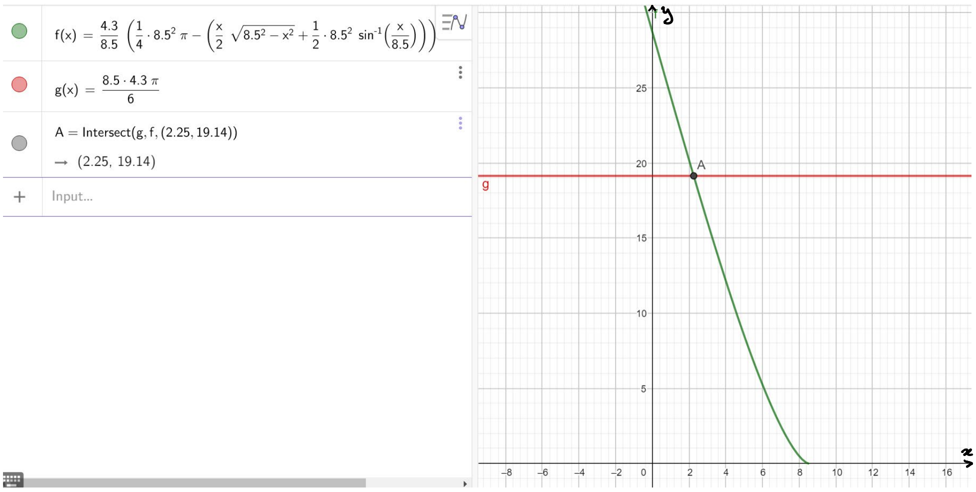

Figure 15 A screen shot of GeoGebra to show the intersection of the graphs

This shows that n≈2.25.

I checked that this value was correct by inputting the definite integral $\int_{2.25}^{8.5} 4.3\sqrt{(1-\frac{x^2}{8.5^2})}\,dx$ into my calculator and this gave me the correct result.

My calculator returns the value 19.1 which is close to $\frac{8.5\times4.3}{6}\pi$ meaning I have the correct result.

I then checked that this was also the case for the other two areas which are $\int_{-2.25}^{2.25} 4.3\sqrt{(1-\frac{x^2}{8.5^2})}\,dx$ and $\int_{-8.5}^{-2.25} 4.3\sqrt{(1-\frac{x^2}{8.5^2})}\,dx$. My calculator again returns approximately 19.1 for both, which is correct.

$8.5 - 2.25 \approx 6.25$

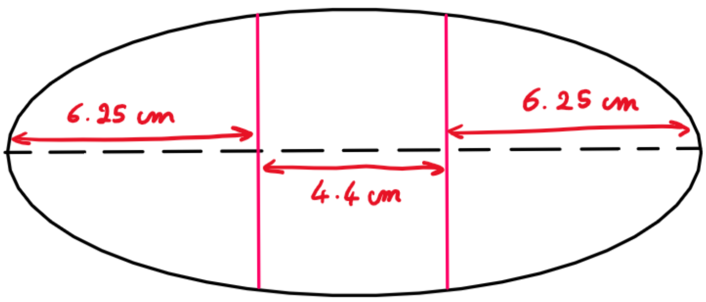

Figure 16 Diagram to show the results of the exploration for the Caprice des Dieux

This means that the cheese slice should be cut 6.25cm along the longest length and the second slice should be cut another 4.4cm along from that.

Conclusion:

Through the use of integration by parts and integration by substitution I have been able to find the area under the graph for a half circle and a half ellipse. Using this method I was then able to find where to cut a Camembert and a Caprice des Dieux into three pieces so that each parallel slice has the same size. The Camembert slices should be 3.75cm, 2.7cm and 3.75cm wide. The Caprice des Dieux slices should be 6.25cm, 4.4cm and 6.25cm wide.

As I worked this out for the general case first I can in fact work this out for any dimensions as all I need to do is replace these in the formula that I found.

At first I was unsure of how to approach the integration by substitution for the circle but then I realised that I had seen a similar example in a previous maths lesson. This gave me a clue as to what substitution to use and allowed me to get past this step and carry on with my integration.

I started with the circle because I thought that this might be the easier shape to work out. In the end the approach and working out for the ellipse turned out to be very similar to that for the circle. This made it easier for me as I had already used this approach once and found that it worked so the second time the steps seemed a lot clearer.

If I were to take this further, I would possibly try to work out a more general formula so that the cheese can be sliced into as many pieces as I choose. Here I limited myself to only three pieces.

Effectively, to do this I would need to again divide the area of the half circle or half ellipse by the number of slices 'm' to gain the area for one slice.

For a circle it would mean working out $\left(\frac{r^2\pi}{4}\right) - \left(\frac{r^2}{2}(\sin^{-1}\frac{a}{r}) + \frac{a\sqrt{r^2-a^2}}{2}\right) = \frac{\pi r^2}{2m}$ and for an ellipse it would be $\frac{b}{a}\left(\frac{1}{4}a^2\pi - \left(\frac{n}{2}\sqrt{a^2-n^2} + \frac{1}{2}a^2\sin^{-1}\frac{n}{a}\right)\right) = \frac{ab}{2m}\pi$. With the circle I would have to find a and then replace r with a and do it again so that I can find the next slice. I would then repeat this process until a became negative. As there is symmetry about the y axis, I would only need to work this out for the $x \geq 0$. For the ellipse I would want to find n and then replace a with n and repeat the process again in the same way as with the circle.